Answer Sheet to Complex Geometry

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1. Let $\Omega \subset \mathbb{C}^n$ be a connected domain. Let $f \in \mathscr{O}(\Omega)$ be a holomorphic function on Ω. If $|f|^2$ is a constant, show that f is constant as well.

   Proof. Let $f = u + \sqrt{-1}v$, then $|f|^2 = f\bar{f} = (u+\sqrt{-1}v)(u-\sqrt{-1}v) = u^2+v^2 \equiv c$. Since $f \in \mathscr{O}(\Omega)$, there holds Cauchy-Riemann equations $\frac{\partial u}{\partial x^i} = \frac{\partial v}{\partial y^i} =: a_i, -\frac{\partial u}{\partial y^i} = \frac{\partial v}{\partial x^i} =: b_i$ for $i = 1, \cdots, n$. Take derivative to $u^2 + v^2 \equiv c$ in the direction of x^i, y^i we have

   $$\begin{cases} u\frac{\partial u}{\partial x^i} + v\frac{\partial v}{\partial x^i} = 0 \\ u\frac{\partial u}{\partial y^i} + v\frac{\partial v}{\partial y^i} = 0 \end{cases},$$

   or,

   $$\begin{cases} a_i u + b_i v = 0 \\ -b_i u + a_i v = 0 \end{cases}.$$

   Solving the equations, we get either $u = v = 0$, which means $f \equiv 0$ is constant, or $a_i^2 + b_i^2 = 0$, which means $a_i = b_i = 0$, leading to that u, v are constant, as well as f. This completes the proof. □

2. Let $\phi, \psi : \mathscr{F} \longrightarrow \mathscr{G}$ be morphisms of sheaves on a topological space $\mathscr{X}$. Show that if $\phi_x = \psi_x$ as maps $\mathscr{F}_x \longrightarrow \mathscr{G}_x$ for all $x \in \mathscr{X}$, then $\phi = \psi$ as morphisms of sheaves.

   Proof. Without lose of generality, one can consider an open $U \subset X$ and a section $f \in \mathscr{F}(U)$. Since $[\phi(f)]_x = \phi_x(f_x) = \psi_x(f_x) = [\psi(f)]_x$ for all $x \in X$, where $(-)_x : \mathscr{F}(U) \longrightarrow \mathscr{F}_x, \mathscr{G}(U) \longrightarrow \mathscr{G}_x$ denotes the natural map, by the definition of passing to the limit, there exists a subset $x \in U_x \subset U$ such that $\phi_{U_x}(f) = \psi_{U_x}(f)$ on $\mathscr{G}(U_x)$. Hence, by the first axiom of sheaf, due to the facts that $U = \bigcup U_x$ as well as that $r^U_{U_x}(\phi(f)) = r^U_{U_x}(\psi(f))$ by the compatibility of sheave morphism with restriction map, there holds $\phi(f) = \psi(f)$ on $\mathscr{G}(U)$. Let f run over all the sections and U all the open subsets to complete the proof. □

3. Let X be a compact complex manifold. Let $\mathscr{O}_X$ be the sheaf of holomorphic functions. Show that $\mathscr{O}_X$ is not a soft sheaf.

   Proof. We can choose suitable K to be closed and to be contained in a suitable chart (U, φ) such that $0 \in \varphi(K)$, then $f \circ \varphi := \sum_{I \in \mathbb{N}^n} z^I$, where I is a multiindex and n the dimension of X, is clearly holomorphic on $K' := K \cap \varphi^{-1}(\overline{B_{1/2}(0)})$. It is well known, however, that a holomorphic function on a compact complex manifold must be a constant. Therefore f cannot be extended to a global holomorphic function on X, i.e. the restriction $\mathscr{O}_X(X) \to \mathscr{O}_X(K')$ is not surjective, violating the definition of soft sheaf. □

4. Let X be a compact complex manifold of dimension n. Let $L_1, \cdots, L_n$ be holomorphic line bundles over X. Assume that L_1 is trivial (i.e. $L_1 \cong X \times \mathbb{C}$). Then show that the intersection number $L_1 \bullet \cdots \bullet L_n = 0$.

   Proof. From definition we have
$$L_1 \bullet \cdots \bullet L_n = \frac{1}{(2\pi)^n} \int_X R_{h_1} \wedge \cdots \wedge R_{h_n}$$
   where $R_h = -\sqrt{-1}\partial\bar{\partial} \log h$. Obviously R_h's are (1,1)-forms, therefore $R_{h_1} \wedge \cdots \wedge R_{h_n}$ exactly gives rise to a volumn form. The triviality of L_1 leads to a globally defined h_1, henceforth $R_{h_1} = \mathrm{d}(\frac{\partial - \bar{\partial}}{2\sqrt{-1}}(\log h_1))$ is d-exact. Applying Stokes' theorem the result is derived. □

5. Show that $S^2 \times S^2$ admits a Kähler structure.

   Proof. Since $S^2 \cong \mathbb{C}P^1$, we can consider $\mathbb{C}P^1 \times \mathbb{C}P^1$ with Fubini-Study metric $(\mathbb{C}P^1, \omega_{\mathrm{FS}}) = (\mathbb{C}P^2, g, J)$. We only need to check that the product of manifolds (X_1, g_1, J_1) and (X_2, g_2, J_2), where X_1, X_2 denote two $\mathbb{C}P^1$ components, exactly give rise to a Kähler structure.

   The product metric on $(X = \mathbb{C}P^1 \times \mathbb{C}P^1, g_0, J_0)$ is given by $g_0(u, v) = g_1(P_1 u, P_1 v) + g_2(P_2 u, P_2 v)$. By letting $J_0 = J_1 \oplus J_2$, the product Kähler form given by $\omega_0(u, v) = \omega_1(P_1 u, P_1 v) + \omega_2(P_2 u, P_2 v)$, where P_- is the projective operator.

   By simple calculation we derive $\omega_0(u, u) = \omega_1(P_1 u, P_1 u) + \omega_2(P_2 u, P_2 u) > 0$ by the positive definiteness of ω_1 and ω_2, meaning that ω_0 is also positive definite. Similarly from $\mathrm{d}\omega_0(u, v) = \mathrm{d}\omega_1(P_1 u, P_1 v) + \mathrm{d}\omega_2(P_2 u, P_2 v) = 0$ the d-closedness is also inherited to the product. Finally by the definition of admitting a Kähler structure, the proof is complete. □