Lecture Notes of Gauge Theory

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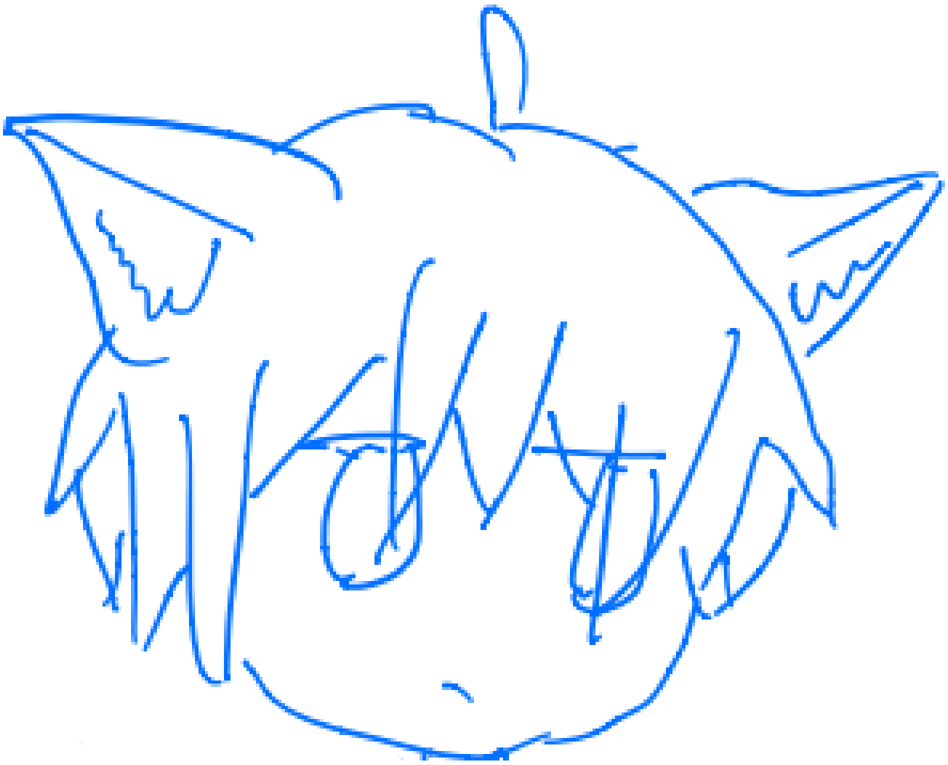
March 22, 2022



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1 Manifold and Bundle

Recall that a **differentiable manifold** of dimension n is nothing other than a family of open sets $\{U_\alpha\}$ together with homeomorphisms $\varphi_\alpha : U_\alpha \cong \mathbb{R}^n$, i.e. locally Euclidean, such that there exist diffeomorphisms $\varphi_\alpha(U_\alpha \cap U_\beta) \approx \varphi_\beta(U_\alpha \cap U_\beta)$ for each pair of α, β.

A **vector field** on a local chart is a map $X_\alpha(-) : U_\alpha \to TU_\alpha$. Since that $TU_\alpha \cong U_\alpha \times \mathbb{R}^n$, it can be expressed via the coordinate as

$$X_\alpha(x) = \{\partial_1^\alpha, \cdots, \partial_n^\alpha\} \begin{bmatrix} \varphi_1^\alpha(x) \\ \vdots \\ \varphi_n^\alpha(x) \end{bmatrix}$$

for $x \in U_\alpha$, where $\partial_i^\alpha := \dfrac{\partial}{\partial x_i^\alpha}$ is the abbreviation. We denote the coordination of X_α by $\psi_\alpha(x) := \begin{bmatrix} \varphi_1^\alpha(x) \\ \vdots \\ \varphi_n^\alpha(x) \end{bmatrix} \in \mathbb{R}^n$. Therefore on $U_\alpha \cap U_\beta$ a vector field X has two coordinate expression w.r.t each chart that should be identical:

$$X(x) = \{\partial_1^\alpha, \cdots, \partial_n^\alpha\} \begin{bmatrix} \varphi_1^\alpha(x) \\ \vdots \\ \varphi_n^\alpha(x) \end{bmatrix} = \{\partial_1^\beta, \cdots, \partial_n^\beta\} \begin{bmatrix} \varphi_1^\beta(x) \\ \vdots \\ \varphi_n^\beta(x) \end{bmatrix}.$$

Recall that $\partial_i^\alpha = \dfrac{\partial}{\partial x_i^\alpha} = \displaystyle\sum_{j=1}^n \frac{\partial x_j^\beta}{\partial x_i^\alpha} \frac{\partial}{\partial x_j^\beta}$, therefore

$$\{\partial_1^\beta, \cdots, \partial_n^\beta\} \begin{bmatrix} \varphi_1^\beta(x) \\ \vdots \\ \varphi_n^\beta(x) \end{bmatrix} = \{\partial_1^\alpha, \cdots, \partial_n^\alpha\} \begin{bmatrix} \varphi_1^\alpha(x) \\ \vdots \\ \varphi_n^\alpha(x) \end{bmatrix} = \{\partial_1^\beta, \cdots, \partial_n^\beta\} \begin{bmatrix} \dfrac{\partial x_j^\beta}{\partial x_i^\alpha} \end{bmatrix}_{n\times n} \begin{bmatrix} \varphi_1^\alpha(x) \\ \vdots \\ \varphi_n^\alpha(x) \end{bmatrix}$$

where j stands for the row and i the column. We derive that

$$\psi_\beta(x) = \begin{bmatrix} \dfrac{\partial x_j^\beta}{\partial x_i^\alpha} \end{bmatrix}_{n\times n} \psi_\alpha(x),$$

as well as

$$\psi_\alpha(x) = \begin{bmatrix} \dfrac{\partial x_j^\alpha}{\partial x_i^\beta} \end{bmatrix}_{n\times n} \psi_\beta(x).$$

If we view the coefficients as observables, it is noteworthy that in different chart we have different result. However, we have already known that each pair of results differs only by a transition matrix. In physics we call the phenomenon as **locality**, which means a family of map $\{\psi_\alpha : U_\alpha \to \mathbb{R}^k\}$ satisfying $[\psi_\alpha(x)]_{k\times 1} = [g_{\alpha\beta}(x)]_{k\times k}[\psi_\beta(x)]_{k\times 1}$ for $x \in U_\alpha \cap U_\beta$. We call $g_{\alpha\beta}(x)$ as **transition function**. In fact the $\{\psi_\alpha : U_\alpha \to \mathbb{R}^k\}$ defined above is **a section of a vector bundle** with the transition functions $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{R}^k\}$. We make a formal definition as follows.

Definition 1.1. *A vector bundle of rank k over a smooth manifold M is a triple $(E \xrightarrow{\pi} M)$ satisfying that*

- *there exists a local trivialization ($M = \bigcup U_\alpha$)*

$$\begin{tikzcd} E|_{U_\alpha} \arrow[rr, "\varphi_\alpha", "\cong"'] \arrow[dr, "\pi"'] & & U_\alpha \times \mathbb{R}^k \arrow[dl] \\ & U_\alpha & \end{tikzcd}$$

- *moreover, the trivialization is a linear isomorphism at each fibre $\varphi_\alpha|_{E_x := \pi^{-1}(x)} : E_x \to \{x\} \times \mathbb{R}^k$.*

Remark 1.2. *It is easy to realize a local trivialization on a bundle. In fact, we can choose a local frame $s_1^\alpha, \cdots, s_k^\alpha : U_\alpha \to E|_{U_\alpha}$ such that $\{s_i^\alpha(x)\}$ is a basis of E_x. Therefore, for each $e \in E_x = \operatorname{span}\{s_1^\alpha, \cdots, s_k^\alpha\}$, we have the expression*

$$e = \{s_1^\alpha(e), \cdots, s_k^\alpha(e)\} \begin{bmatrix} \varphi_1^\alpha(e) \\ \vdots \\ \varphi_k^\alpha(e) \end{bmatrix},$$

and the trivialization is hence realized as

$$\varphi_\alpha(e) = (\pi(e), \begin{bmatrix} \varphi_1^\alpha(e) \\ \vdots \\ \varphi_k^\alpha(e) \end{bmatrix}).$$

When $x \in U_\alpha \cap U_\beta$, we have two local frames $\{s_i^\alpha\}, \{s_j^\beta\}$. Similarly we have

$$\{s_1^\alpha(x), \cdots, s_k^\alpha(x)\} = \{s_1^\beta(x), \cdots, s_k\beta(x)\} g_{\alpha\beta}(x),$$

where $g_{\alpha\beta}(-) : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$, as well as

$$\begin{bmatrix} \varphi^\beta(x) \\ \vdots \\ \varphi^\beta(x) \end{bmatrix} = g_{\alpha\beta} \begin{bmatrix} \varphi_1^\alpha(x) \\ \vdots \\ \varphi_k^\alpha(x) \end{bmatrix}.$$

When $x \in U_\alpha \cap U_\beta \cap U_\gamma$, we can find out that

$$g_{\alpha\beta}(x)g_{\beta\gamma}(x) = g_{\alpha\gamma}(x).$$

We call it the ***1-cocycle condition for transition functions****.*

Remark 1.3. *Reconstruction Theorem. Given the data of local trivializations as well as transition functions, we can reconstruct the bundle up to an isomorphism, i.e.*

$$E \cong \frac{\bigsqcup_\alpha U_\alpha \times \mathbb{R}^k}{(x,v)_\beta \sim (x, g_{\alpha\beta}(x)v)_\alpha}.$$

Here by bundle isomorphism $E_1 \cong E_2$ we mean a diffeomorphism $\psi : E_1 \approx E_2$ such that the diagram

$$\begin{tikzcd} E_1 \arrow[rr, "\psi", "\approx"'] \arrow[dr, "\pi_1"'] & & E_2 \arrow[dl, "\pi_2"] \\ & M & \end{tikzcd}$$

commutes as well as it is a fiberwisely linear isomorphism.

Definition 1.4. *A* ***section*** *is a map $s : M \to E$ such that $\pi \circ s = id$.*

In local trivialization, a section s is corresponding to the family $\{\varphi_\alpha : U_\alpha \to \mathbb{R}^k\}$ satisfying $\varphi_\alpha(x) = g_{\alpha\beta}(x)\varphi_\beta(x)$.

We denote the space of sections as

$$\Gamma(E) = \{s : M \to E | \pi \circ s = id_M\}.$$

Remark 1.5. $\Gamma(E)$ *could admit several completion such as L^2-section or L^2_k-section, where L^p_k is Sobolev norm.*

And we make $\mathfrak{X}(M) = \Gamma(TM)$ into a module over $C^\infty(M)$.

Definition 1.6. *A* ***connection(covariant derivative)*** *is a map*

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E) : (X, s) \mapsto \nabla_X s$$

satisfying

1. *$C^\infty(M)$-linear in X: $\nabla_{fX} s = f\nabla_X s$ for $f \in C^\infty(M)$;*

2. *$\mathbb{R}$-linear in s;*

3. *Leibniz rule:* $\nabla_X(fs) = X(f)s + f\nabla_X s$.

Example 1.7. *Trivial bundle & its trivial connection. We call a bundle with the form* $E = M \times \mathbb{R}^k$ *a trivial bundle. Every section* $s \in \Gamma(E) = C^\infty(M, \mathbb{R}^k)$ *is a* $\mathbb{R}^k$*-value function. Therefore we can define a so-called* ***trivial connection*** *as follows*

$$\nabla^{tri} : (X, \begin{bmatrix} s_1 \\ \vdots \\ s_k \end{bmatrix}) \mapsto \begin{bmatrix} Xs_1 \\ \vdots \\ Xs_k \end{bmatrix}.$$

Example 1.8. *It is not generally true that there is a trivial connection for any bundle in the following sense:*

$$\nabla^{tri}|_{U_\alpha} : \mathfrak{X}(U_\alpha) \times C^\infty(U_\alpha, \mathbb{R}^k) \to C^\infty(U_\alpha, \mathbb{R}^k) : (X, \psi^\alpha = \begin{bmatrix} \psi_1^\alpha(x) \\ \vdots \\ \psi_k^\alpha(x) \end{bmatrix}) \mapsto X\psi^\alpha = \begin{bmatrix} X\psi_1^\alpha(x) \\ \vdots \\ X\psi_k^\alpha(x) \end{bmatrix}.$$

In fact, by the transition function for $x \in U_\alpha \cap U_\beta$ *we have* $\psi^\beta = g_{\alpha\beta}\psi^\alpha$*. Take derivative we have* $X\psi^\beta = g_{\alpha\beta}X\psi^\alpha$ *and by the Leibniz rule we derive* $Xg_{\alpha\beta} = 0$*. We can conclude that* ∇^{tri} *exists if and only if* $g_{\alpha\beta}$ *is locally constant on* $U_\alpha \cap U_\beta$*. That is, if* $g_{\alpha\beta}$ *were not locally constant, then these local trivial connection would not define a connection on* E*.*

Since trivial connections are not always defined a global connection, we may wonder that

Question: Is the space of connection empty?

No. Via the partition of unity $\{\rho_\alpha : U_\alpha \to \mathbb{R}\}$ with $\sum_\alpha \rho_\alpha(x) = 1$ for $x \in M$, we can define $\nabla_X s = \sum_\alpha X(\rho_\alpha s)$, which is indeed a global connection.

2 Endomorphism Bundle Valued 1-Form

Last time we have known that the space of connections is not empty. In fact,

Proposition 2.1. *The space of connections $\mathscr{A}$ is an affine space modelled on $\Omega^1(M, \operatorname{End} E)$.*

By affine space we mean that for a fixed $A_0 \in \mathscr{A}$ we have $\mathscr{A} = A_0 + \Omega^1(M, \operatorname{End} E)$, in other words, $A - A_0 \in \Omega^1(M, \operatorname{End} E)$ for all $A \in \mathscr{A}$.

Today we are going to explain in detail what $\Omega^1(M, \operatorname{End} E)$, a space of a vector bundle valued 1-form, is.

What is a 1-form? The space of 1-forms is the cotangent bundle $\Omega^1(M) = \Gamma(M, T^*M)$. We have known that on U_α the sections of tangent bundle $\mathfrak{X}(U_\alpha) = \Gamma(TU_\alpha)$ admits a local frame $\{\partial_1^\alpha, \cdots, \partial_n^\alpha\}$. We then take its dual frame $\{\mathrm{d}x_1^\alpha, \cdots, \mathrm{d}x_n^\alpha\}$ such that $\mathrm{d}x_i^\alpha(\partial_j^\alpha) = \delta_{ij}$. So a 1-form is locally an element in the vector space $\operatorname{span}\{\mathrm{d}x_1^\alpha, \cdots, \mathrm{d}x_n^\alpha\} \cong \Omega^1(U_\alpha)$.

Recall that $\Omega^0(U_\alpha) = C^\infty(U_\alpha)$ and that $\Omega^0(U_\alpha) \xrightarrow{\mathrm{d}} \Omega^1(U_\alpha) : f \mapsto \mathrm{d}f = \sum_{x=1}^n \frac{\partial f}{\partial x_i^\alpha}\mathrm{d}x_i^\alpha$. Therefore on $U_\alpha \cap U_\beta$ we have

$$\mathrm{d}x_i^\beta = \sum_{j=1}^n \frac{\partial x_i^\beta}{\partial x_j^\alpha}\mathrm{d}x_j^\alpha.$$

Denoting $h_{\alpha\beta} = \left[\frac{\partial x_i^\beta}{\partial x_j^\alpha}\right]_{ij}$, we derive that

$$\{\mathrm{d}x_1^\alpha, \cdots, \mathrm{d}x_n^\alpha\} \cdot h_{\alpha\beta} = \{\mathrm{d}x_1^\beta, \cdots, \mathrm{d}x_n^\beta\},$$

where $h_{\alpha\beta}$ is the transition matrix for 1-forms.

Remark 2.2. *If $g_{\alpha\beta} = \left[\frac{\partial x_i^\alpha}{\partial x_j^\beta}\right]_{ij}$ is the transition matrix for $\Gamma(TM)$, i.e. $\{\partial_1^\alpha, \cdots, \partial_n^\alpha\} \cdot g_{\alpha\beta} = \{\partial_1^\beta, \cdots, \partial_n^\beta\}$, then we can easily see that $h_{\alpha\beta} = g_{\alpha\beta}^{-1}$.*

What is a vector bundle valued 1-form? We denote the space of such 1-forms as $\Omega^1(M, E)$. Locally we have $E|_{U_\alpha} \cong U_\alpha \times \mathbb{R}^k$, where the isomorphism is w.r.t. a local frame $\{s_1, \cdots, s_k\}$. Hence *mutatis mutandis* a vector bundle valued 1-form is locally $\omega^\alpha = \begin{bmatrix} \omega_1^\alpha \\ \vdots \\ \omega_k^\alpha \end{bmatrix} \in \Omega^1(M, \mathbb{R}^k)$, where $\omega_i^\alpha \in \Omega^1(M, \mathbb{R})$, with transition function $g_{\alpha\beta} \in \mathrm{GL}(k, \mathbb{R})$ s.t. $\omega^\alpha = g_{\alpha\beta} \cdot \omega^\beta$. (*Nota bene.* This time $g_{\alpha\beta}$ is the transition matrix for E instead for $\Gamma(TM)$.)

To conclude, $\Omega^1(M, E) = \{(\omega_\alpha \in \Omega^1(U_\alpha, \mathbb{R}^k))_\alpha : \omega^\alpha = g_{\alpha\beta}\omega^\beta\}$. We can define $\Omega^k(M, E)$ following the above fashion.

What is an endomorphism bundle $\operatorname{End} E$**?** Naturally we have $\operatorname{End} E = \bigsqcup_x \operatorname{End} E_x$. Since there holds $E_x \cong \mathbb{R}^k$ w.r.t. a local frame $\{s_1, \cdots, s_k\}$ on U_α, we have $\operatorname{End} E_x \cong M_k(\mathbb{R}) = \mathrm{gl}_k(\mathbb{R})$ as a fibre. That is, on U_α we have $\operatorname{End} E|_{U_\alpha} \cong U_\alpha \times \mathrm{gl}_k(\mathbb{R})$. We only need to find out the transition function.

Remark 2.3. $\mathrm{gl}_k(\mathbb{R})$ *is the Lie algebra of* $\mathrm{GL}(k, \mathbb{R})$*, with Lie bracket defined as* $[A, B] = AB - BA$ *for* $A, B \in \mathrm{gl}_k(\mathbb{R})$*. Firstly* $\mathrm{gl}_k(\mathbb{R})$ *is isomorphism as vector space to* $T_I \mathrm{GL}(k, \mathbb{R})$*. Indeed, since* $\mathrm{GL}(k, \mathbb{R})$ *is an open subset of* $M_k(\mathbb{R})$*, it has the same tangent space as latter's:* $T_I \mathrm{GL}(k, \mathbb{R}) = M_k(\mathbb{R})$*. Then we can view* $T_I \mathrm{GL}(k, \mathbb{R})$ *as the left invariant vector fields of* $\mathrm{GL}(k, \mathbb{R})$*. In fact, let* ξ_A, ξ_B *be the left invariant vector fields generated by* $A, B \in T_I \mathrm{GL}(k, \mathbb{R})$*, we have* $[\xi_A, \xi_B] = \xi_{[A,B]}$*.*

Notice that $\operatorname{End} E_x = E_x^* \otimes E_x$, taking the dual frame $\{t^1, \cdots, t^k\}$ such that $t^j(s_i) = \delta_i^j$, there holds $\operatorname{End} E_x \cong \operatorname{span}\{s_i \otimes t^j\}$. A section $\xi \in U_\alpha$ has an expression as

$$\xi = \{s_1^\alpha, \cdots, s_k^\alpha\} \left[a_j^i\right] \begin{Bmatrix} t_\alpha^1 \\ \vdots \\ t_\alpha^k \end{Bmatrix}.$$

Let's denote the matrix $\left[a_j^i\right]$ as A^α. When $\xi \in U_\alpha \cap U_\beta$, we have

$$\xi = \{s_1^\alpha, \cdots, s_k^\alpha\} A^\alpha \begin{Bmatrix} t_\alpha^1 \\ \vdots \\ t_\alpha^k \end{Bmatrix} = \{s_1^\beta, \cdots, s_k^\beta\} A^\beta \begin{Bmatrix} t_\beta^1 \\ \vdots \\ t_\beta^k \end{Bmatrix} = \{s_1^\alpha, \cdots, s_k^\alpha\} g_{\alpha\beta} A^\beta g_{\alpha\beta}^{-1} \begin{Bmatrix} t_\alpha^1 \\ \vdots \\ t_\alpha^k \end{Bmatrix},$$

that is,

$$A^\alpha = g_{\alpha\beta} A^\beta g_{\alpha\beta}^{-1} = \mathrm{Ad}_{g_{\alpha\beta}^{-1}} A^\beta,$$

where $\mathrm{Ad}_{g_{\alpha\beta}^{-1}} : U_\alpha \cap U_\beta \to \mathrm{GL}(\mathrm{gl}_k(\mathbb{R})) : x \mapsto (A \mapsto g_{\alpha\beta}(x) A g_{\alpha\beta}^{-1}(x))$ is our desired transition function for $\operatorname{End} E$.

$$\Gamma(M, \operatorname{End} E) = \{(A^\alpha : U_\alpha \to \mathrm{gl}_k \mathbb{R})_\alpha : A^\alpha = \mathrm{Ad}_{g_{\alpha\beta}^{-1}} A^\beta \text{ on } U_\alpha \cap U_\beta\}.$$

Therefore,

$$\Omega^1(M, \operatorname{End} E) = \{(\omega_\alpha \in \Omega^1(U_\alpha, \mathrm{gl}_k(\mathbb{R})))_\alpha : \omega_\alpha = g_{\alpha\beta} \omega_\beta g_{\alpha\beta}^{-1} \text{ on } U_\alpha \cap U_\beta\}.$$

3 The Space of Connections, Principal Bundles

Today we are going to prove the proposition we stated before.

Proposition 3.1 (Restatement.). *The space of connections $\mathscr{A}$ is an affine space modelled on* $\Omega^1(M, \operatorname{End} E)$.

Proof. 1. **The local expression of a connection.** Recall that

$$\begin{array}{cccc} \nabla: & \mathfrak{X}(M)\times & \Gamma(E) \to & \Gamma(E) \\ (& X, & s) \mapsto & \nabla_X s. \end{array}$$

On U_α, we have $E|_{U_\alpha} \cong U_\alpha \times \mathbb{R}^k$ w.r.t. a local frame $\{\sigma_1, \cdots, \sigma_k\}$. Along with $U_\alpha \cong \mathbb{R}^n$ w.r.t. $\{\partial_1, \cdots, \partial_n\}$, we have

$$X = \sum_{i=1}^{n} X_i \partial_i, \quad s = \sum_{j=1}^{k} \psi_j \sigma_j.$$

By simple calculation,

$$\begin{aligned} \nabla_X s &= \sum_{i=1}^{n} X_i \nabla_{\partial_i}(\{\sigma_1, \cdots, \sigma_k\} \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_k \end{bmatrix}) \\ &= \sum_{i=1}^{n} X_i \left((\nabla_{\partial_i}\{\sigma_1, \cdots, \sigma_k\}) \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_k \end{bmatrix} + \{\sigma_1, \cdots, \sigma_k\} \begin{bmatrix} \partial_i\psi_1 \\ \vdots \\ \partial_i\psi_k \end{bmatrix} \right). \end{aligned}$$

Denoting $\nabla_{\partial_i}\{\sigma_1, \cdots, \sigma_k\} = \{\sigma_1, \cdots, \sigma_k\} A_i$, where $A_i \in \mathrm{GL}(k, \mathbb{R})$, we can define the so-called **local connection 1-form** as

$$A = A_i \mathrm{d}x_i \in \Omega^1(U_\alpha, \mathrm{gl}_k(\mathbb{R})),$$

therefore

$$\nabla\{\sigma_1, \cdots, \sigma_k\} = \{\sigma_1, \cdots, \sigma_k\} A.$$

Noticing that $\nabla_{\partial_i}\psi = \partial_i\psi = \mathrm{d}\psi(\partial_i)$, we can write

$$\nabla s = \{\sigma_1, \cdots, \sigma_k\} A\psi + \{\sigma_1, \cdots, \sigma_k\} \mathrm{d}\psi,$$

or,

$$\nabla = A + \mathrm{d}.$$

2. **The transition function of a connection.** Remember that on $U_\alpha \cap U_\beta$ we have $\{\sigma_1^\alpha, \cdots, \sigma_k^\alpha\}g_{\alpha\beta} = \{\sigma_1^\beta, \cdots, \sigma_k^\beta\}$. And from the above result we have known that

$$\nabla\{\sigma_1^\alpha, \cdots, \sigma_k^\alpha\} = \{\sigma_1^\alpha, \cdots, \sigma_k^\alpha\}A^\alpha, \quad \nabla\{\sigma_1^\beta, \cdots, \sigma_k^\beta\} = \{\sigma_1^\beta, \cdots, \sigma_k^\beta\}A^\beta.$$

Therefore, on the one hand

$$\begin{aligned}\nabla\{\sigma_1^\beta, \cdots, \sigma_k^\beta\} &= \nabla(\{\sigma_1^\alpha, \cdots, \sigma_k^\alpha\}g_{\alpha\beta}) \\ &= (\nabla\{\sigma_1^\alpha, \cdots, \sigma_k^\alpha\})g_{\alpha\beta} + \{\sigma_1^\alpha, \cdots, \sigma_k^\alpha\}\nabla g_{\alpha\beta} \\ &= \{\sigma_1^\alpha, \cdots, \sigma_k^\alpha\}A^\alpha g_{\alpha\beta} + \{\sigma_1^\alpha, \cdots, \sigma_k^\alpha\}\mathrm{d}g_{\alpha\beta}.\end{aligned}$$

On the other hand,

$$\{\sigma_1^\beta, \cdots, \sigma_k^\beta\}A^\beta = \{\sigma_1^\alpha, \cdots, \sigma_k^\alpha\}g_{\alpha\beta}A^\beta.$$

To sum up, we derive that

$$g_{\alpha\beta}A^\beta = A^\alpha g_{\alpha\beta} + \mathrm{d}g_{\alpha\beta},$$

i.e.

$$A^\beta = g_{\alpha\beta}^{-1}A^\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1}\mathrm{d}g_{\alpha\beta},$$

or,

$$A^\alpha = g_{\alpha\beta}A^\beta g_{\alpha\beta}^{-1} - \mathrm{d}g_{\alpha\beta} \cdot g_{\alpha\beta}^{-1}.$$

We call the last formula **local gauge transformation law**.

3. **Relationship between 2 connections.** Suppose we have two connections A_0, A_1 with their local connection 1-forms $\{A_0^\alpha\}, \{A_1^\alpha\}$. On $U_\alpha \cap U_\beta$, obviously,

$$A_0^\alpha - A_1^\alpha = \mathrm{Ad}_{g_{\alpha\beta}^{-1}}(A_0^\beta - A_1^\beta).$$

This implies that

$$\{(A_0^\alpha - A_1^\alpha \in \Omega^1(U_\alpha, \mathrm{gl}_k(\mathbb{R})))_\alpha\}$$

defines a global section of $T^*M \otimes \mathrm{End}\,E$, i.e. the $\mathrm{End}\,E$ valued 1-form.

□

We are going to look at an example of what we later will call a **principal bundle**.

Given a vector bundle E over M of rank k, we define the **frame bundle** as a fibre bundle $\mathcal{F}r_{\mathrm{GL}_k}(E) \longrightarrow M$, where $\mathcal{F}r_{\mathrm{GL}_k}(E) = \bigcup_{x\in M}\{\text{frames of } E_x\} = \bigcup_{x\in M}\mathcal{F}r_{\mathrm{GL}_k}(E_x)$.

At each fibre, $\mathcal{F}r_{\mathrm{GL}_k}(E_x) = \{\text{bases of } E_x\}$ admits a free transitive right action on $\mathrm{GL}(k,\mathbb{R})$, which means for some frame $\sigma(x) = \{\sigma_1(x), \cdots, \sigma_k(x)\} \in \mathcal{F}r_{\mathrm{GL}_k}(E_x)$, we have $\mathcal{F}r_{\mathrm{GL}_k}(E_x) = \sigma(x) \cdot \mathrm{GL}(k,\mathbb{R})$, as well as $\exists \sigma(x) : \sigma(x) \cdot g = \sigma(x) \implies g = I$.

Since $E|_{U_\alpha} \cong U_\alpha \times \mathbb{R}^k$, the trivialization of the frame bundle is realized as

$$\mathcal{F}r_{\mathrm{GL}_k}(E|_{U_\alpha}) \cong U_\alpha \times \mathrm{GL}(k,\mathbb{R}).$$

Similarly we can define

Definition 3.2. *A* ***principal*** G***-bundle*** P *over a manifold* M *is a local trivializable fibre bundle such that each fibre admits a free transitive right* G*-action. We require* G *to be a Lie group. We call* G *the* ***structure group*** *of* P*.*

Example 3.3. $\mathcal{F}r_{\mathrm{GL}_k}(E) \longrightarrow M$ *is a principal* $\mathrm{GL}(k,\mathbb{R})$*-bundle.*

Example 3.4. *If we equip the frame bundle with a fiberwise inner product, making it into a* ***Euclidean vector bundle****, then* $\bigcup\{\textit{orthonormal bases of } E_x\} = \mathcal{F}r_{\mathrm{O}(k)}(E) \longrightarrow M$ *is a principal* $\mathrm{O}(k)$*-bundle.*

Remark 3.5. *The structure group of* $\mathcal{F}r_{\mathrm{GL}_k}(E)$ *can be reduced to* $\mathrm{O}(k)$*, but not to* $\mathrm{SO}(k)$*.*

Example 3.6. *Moreover, if* E *is orientable, then* $\bigcup\{\textit{oriented orthonormal bases of } E_x\} = \mathcal{F}r_{\mathrm{SO}(k)}(E) \longrightarrow M$ *is a principal* $\mathrm{SO}(k)$*-bundle.*

Definition 3.7. *Given a principal* G*-bundle* P *and a linear representation of* G *on a vector space* $V \cong \mathbb{R}^k$

$$\rho : G \to \mathrm{GL}(V) \cong \mathrm{GL}(k,\mathbb{R}),$$

then there exists an ***associated vector bundle***

$$P \times_\rho V := \frac{P \times V}{(p,v) \sim (p \cdot g^{-1}, \rho(g) \cdot v)}.$$

An ***adjoint bundle*** *is defined as*

$$\mathrm{ad}\, P = P \times_{\mathrm{ad}} \mathfrak{g},$$

where $\mathrm{ad} : G \to \mathrm{GL}(\mathfrak{g}) : g \mapsto \mathrm{ad}_g$*, where we let* $\exp \mathrm{ad}_g(h) \equiv g \exp(h) g^{-1}$*.*

Example 3.8. $\mathrm{End}\, E = \mathrm{ad}\, \mathcal{F}r_{\mathrm{GL}_k}(E)$.

4 Connection on Principal Bundle, Curvature

Recall that a connection on a bundle $E \longrightarrow M$ consists of local connection 1-forms

$$\{A^\alpha \in \Omega^1(U_\alpha, \mathrm{gl}_k \mathbb{R})\}$$

satisfying the local gauge transformation law

$$A^\alpha = \mathrm{ad}_{g_{\alpha\beta}} A^\beta - \mathrm{d}g_{\alpha\beta} \cdot g_{\alpha\beta}^{-1}.$$

Moreover, the space of connections is an affine space modelled on $\Omega^1(M, \mathrm{End}\, E)$.

In fact, we have the following result of what we will define later as a connection on a principal bundle. Similarly, the connection on a G-principal bundle P consists of local connection 1-forms

$$\{A^\alpha \in \Omega^1(U_\alpha, \mathfrak{g})\}$$

satisfying the local gauge transformation law

$$A^\alpha = \mathrm{ad}_{g_{\alpha\beta}} A^\beta - \mathrm{d}g_{\alpha\beta} \cdot g_{\alpha\beta}^{-1}.$$

as well. Notice that $\mathrm{ad}\, \mathcal{F}r_{\mathrm{GL}_k} E = \mathrm{End}\, E$, the space of connections on a principal bundle is an affine space modelled on $\Omega^1(M, \mathrm{ad}\, P)$.

Now we will give 3 equivalent definitions of a connection on a principal bundle $P \longrightarrow M$, though we will not show the equivalence here.

The first way. A connection on $P \xrightarrow{\pi} M$ is a G-invariant decomposition

$$TP \cong T^{\mathrm{Vert}} P \oplus \pi^* TM,$$

where π^*TM is the pullback bundle of $TM \longrightarrow M$ via $P \xrightarrow{\pi} M$, namely $\pi^*TM = \{(p, e) \in P \times TM \mid \pi(p)$

Remark 4.1. *Using the language of category theory, a pullback bundle is literally the pullback of the diagram $P \xrightarrow{\pi} M \leftarrow TM$ up to an isomorphism, i.e. the diagram*

$$\begin{tikzcd}
Q \arrow[dr, dashed, "\exists!"] \arrow[drr] \arrow[ddr] & & \\
& \pi^*TM \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & TM \arrow[d] \\
& P \arrow[r, "\pi"'] & M
\end{tikzcd}$$

commutes for all Q.

We call $T^{\mathrm{Vert}}P$ the **verticle bundle**. It is easy to see that $T_p^{\mathrm{Vert}}P = T_pP_{\pi(p)}$, the tangent space at p on the fiber $P_{\pi(p)} = \pi^{-1}(\pi(p))$. Notice that an infinitsimal action of $\mathfrak{g} = \mathrm{Lie}\, G$ of P is defined as a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{X}(M)$ that we associate to $\xi \in \mathfrak{g}$ a **vertical vector field**

$$\tilde{\xi}(p) = [p\exp(t\xi)] = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (p\exp(t\xi)) .$$

Choosing a basis $\{\xi_1, \cdots, \xi_k\}$ of $\mathfrak{g}$, where $k = \dim \mathfrak{g}$, we have that $\{\tilde{\xi}_1, \cdots, \tilde{\xi}_k\}$ is a global frame of $T^{\mathrm{Vert}}P$ due to the freeness and transitiveness of the action. Therefore we find out that

$$T^{\mathrm{Vert}}P \cong P \times \mathfrak{g} =: \underline{\mathfrak{g}}$$

is a trivial bundle.

The direct sum decomposition is amount to say that the short exact sequence

$$0 \longrightarrow T^{\mathrm{Vert}}P \longrightarrow TP \longrightarrow \pi^*TM \longrightarrow 0$$

is split. Hence there is a horizontal lifting $\tilde{\bullet} : TM \to TP$ such that $\mathrm{d}\pi|_p\tilde{\eta}(p) = \pi^*\eta(\pi(p))$ as well as G-invariant, i.e. $(R_g)_*\tilde{\eta}(p) = \tilde{\eta}(p \cdot g)$.

Remark 4.2. *Not every exact sequence is split. A non-example is given as*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \xrightarrow{\mathrm{mod}\ m} \mathbb{Z}_m \longrightarrow 0,$$

which is exact but not split.

The second way. To make the above short exact sequence split, we can take a G-invariant $\theta : TP \to T^{\mathrm{Vert}}P$ as a **connection** such that $\theta(\tilde{\xi}) = \xi$ for $\tilde{\xi}$ being the vertical vector field generated by $\xi \in \mathfrak{g}$. Since $T^{\mathrm{Vert}}P \cong P \times \mathfrak{g}$, we have $\theta \in \Omega^1(P, \mathfrak{g})$. By G-invariant we mean that for any right translation $R_g : P \to P : p \mapsto pg, g \in G$, we have $R_g^*\theta = \mathrm{ad}_{g^{-1}}\, \theta$, i.e. the diagram

$$\begin{tikzcd} T_pP \arrow[r, "\theta"] \arrow[d, "R_{g*}"'] \arrow[dr, "R_g^*\theta"] & \mathfrak{g} \arrow[d, "\mathrm{ad}_{g^{-1}}"] \\ T_{pg}P \arrow[r, "\theta"'] & \mathfrak{g} \end{tikzcd}$$

commutes.

Locally, we can choose a section $\sigma_\alpha : U_\alpha \to P$. The local connection 1-form of θ on U_α is defined as $\theta_\alpha = \sigma_\alpha^*\theta \in \Omega^1(U_\alpha, \mathfrak{g})$. Notice that on $x \in U_\alpha \cap U_\beta$ we have that $\sigma_\alpha(x) \cdot g_{\alpha\beta}(x) = \sigma_\beta(x)$.

By calculation, letting $\gamma : (-\epsilon, \epsilon) \to U_\alpha \cap U_\beta$ with $\gamma(0) = x, \gamma'(0) = X$, at $\sigma_\beta(x) \in P$,

$$
\begin{aligned}
\theta_\beta(X) = \sigma_\beta^*\theta(X) = \theta\sigma_{\beta*}X = \theta \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\sigma_\beta(\gamma(t))) &= \theta \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\sigma_\alpha(\gamma(t))g_{\alpha\beta}(\gamma(t))) \\
&= \theta \left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\sigma_\alpha(\gamma(t))g_{\alpha\beta}(x)) + \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\sigma_\alpha(x)g_{\alpha\beta}(\gamma(t))) \right) \\
&= \theta \left(R_{g_{\alpha\beta}*}\sigma_{\alpha*}X + \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\sigma_\beta(x) \exp \left(t \cdot L_{g_{\alpha\beta}^{-1}(x)*}\mathrm{d}g_{\alpha\beta}X \right) \right) \right) \\
&= \mathrm{ad}_{g_{\alpha\beta}^{-1}} \theta_\alpha X + g_{\alpha\beta}^{-1}\mathrm{d}g_{\alpha\beta}X,
\end{aligned}
$$

i.e.

$$
\theta_\beta = \mathrm{ad}_{g_{\alpha\beta}^{-1}} \theta_\alpha + g_{\alpha\beta}^{-1}\mathrm{d}g_{\alpha\beta},
$$

or,

$$
\theta_\alpha = \mathrm{ad}_{g_{\alpha\beta}} \theta_\beta - \mathrm{d}g_{\alpha\beta} \cdot g_{\alpha\beta}^{-1},
$$

exactly the local gauge transformation law. All the θ_α's give rise to **the third way** of defining a connection on a principal bundle.

To show the equivalence between the second and the third way of defining a connection, it is sufficient to show that θ_α's can recover a global 1-form $\theta \in \Omega^1(P, \mathfrak{g})$ on P. Let $\theta^\alpha \in \Omega^1(P|_{U_\alpha}, \mathfrak{g})$ satisfy

$$
\theta^\alpha(\sigma_{\alpha*}X + \tilde{\xi}) = \theta_\alpha X + \xi
$$

as well as G-invariant

$$
\theta^\alpha R_{g*} = R_g^*\theta^\alpha = \mathrm{ad}_{g^{-1}} \theta^\alpha.
$$

We claim that $\theta^\alpha = \theta^\beta$ on $P|_{U_\alpha \cap U_\beta}$, then they will give rise to a global 1-form, namely there will hold $\theta|_{\sigma(U_\alpha)} = \theta^\alpha$. Suppose $\gamma(0) = x \in U_\alpha \cap U_\beta, \gamma'(0) = X$. Then following the above calculation, we have

$$
\begin{aligned}
\theta^\alpha(\sigma_{\beta*}X) &= \theta^\alpha \left(R_{g_{\alpha\beta}*}\sigma_{\alpha*}X + \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\sigma_\beta(x) \exp(t \cdot L_{g_{\alpha\beta}^{-1}(x)*}\mathrm{d}g_{\alpha\beta}(X)) \right) \right) \\
&= \mathrm{ad}_{g_{\alpha\beta}^{-1}} \theta_\alpha(X) + g_{\alpha\beta}^{-1}\mathrm{d}g_{\alpha\beta}(X) = \theta_\beta X.
\end{aligned}
$$

Meanwhile, $\theta^\beta(\sigma_{\beta*}X) = \theta_\beta(X)$. Our claim has been proved. It is obvious that $\theta_\alpha = \sigma_\alpha^*\theta$, where θ is defined as above.

Remark 4.3. *The 1-form in coordination $\Theta = \mathrm{d}g \cdot g^{-1} \in \Omega^1(G, \mathfrak{g})$ is called the* ***Mawer-Cartan form****. There holds $\Theta(\tilde{\xi}) = \xi$, where $\tilde{\xi}$ is a left invariant vector field generated by $\xi \in \mathfrak{g}$.*

Given a connection on a principal G-bundle P over M $\{\theta^\alpha \in \Omega^1(U_\alpha, \mathfrak{g})\}_\alpha$ satisfying the local gauge transformation law $\theta^\alpha = \mathrm{ad}_{g_{\alpha\beta}} \theta^\beta - \mathrm{d}g_{\alpha\beta} \cdot g_{\alpha\beta}^{-1}$ as well as any associated bundle for a linear representation $\rho : G \to \mathrm{GL}(V)$, we have an induced connection on $P \times_\rho V$, namely $\{A_\rho^\alpha = \mathrm{d}\rho_e(\theta^\alpha) \in \Omega^1(U_\alpha, \mathrm{gl}\, V)\}_\alpha$, satisfying the local gauge transformation law $A_\rho^\alpha = \mathrm{ad}_{\rho(g_{\alpha\beta})} A_\rho^\beta - \mathrm{d}\rho(g_{\alpha\beta}) \cdot \rho(g_{\alpha\beta})^{-1}$.

Example 4.4. *We have a connection on a vector bundle*

$$\nabla : \mathfrak{X}(M) \times \Gamma(M, E) \to \Gamma(M, E) \iff \Omega^0(M, E) \to \Omega^1(M, E),$$

which naturally gives rise to connection on the frame bundle

$$\nabla : \Omega^0(M, \mathcal{F}r_{\mathrm{GL}} E) \to \Omega^0(M, \mathcal{F}r_{\mathrm{GL}} E))$$

since their local connection 1-forms both lies in $\Omega^1(U_\alpha, \mathrm{gl}\, \mathbb{R}^k)$, then we can get an induced connection on

$$\nabla^\rho : \Omega^0(M, \mathcal{F}r_{\mathrm{GL}} E \times_\rho V) \to \Omega^1(M, \mathcal{F}r_{\mathrm{GL}} E \times_\rho V).$$

Remark 4.5. *We can induce connections on new bundles. Suppose that ∇^{E_i} are connections on $E_i, i = 1, 2$. We have*

$$\nabla^{E_1 \oplus E_2}(s_1, s_2) = (\nabla^{E_1} s_1, \nabla^{E_2} s_2),$$

$$\nabla^{E_1 \otimes E_2}(s_1 \otimes s_2) = \nabla^{E_1} s_1 \otimes s_2 + s_1 \otimes \nabla^{E_2} s_2.$$

We now define what the curvature of a given connection is.

Definition 4.6. *The curvature of a connection ∇ is defined by*

$$F_\nabla(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]},$$

for $X, Y \in \mathfrak{X}(M)$.

Proposition 4.7. *$F_\nabla \in \Omega^2(M, \mathrm{End}\, E)$, i.e. locally, $F_\alpha := F_\nabla|_{U_\alpha} \in \Omega^2(U_\alpha, \mathrm{gl}_k \mathbb{R})$ satisfying $F_\alpha = \mathrm{ad}_{g_{\alpha\beta}} F_\beta$ on $U_\alpha \cap U_\beta$.*

Proof. Notice that $[\partial_i, \partial_j] = 0$. Suppose that ∇ is given by connection 1-forms $\{A^\alpha =$

$\sum_{i=1}^n A_i^\alpha \mathrm{d}x_i^\alpha\}$ satisfying local gauge transformation law. Hence on U_α,

$$\begin{aligned} F_\alpha(\partial_i^\alpha, \partial_j^\alpha) = F_\nabla|_{U_\alpha}(\partial_i^\alpha, \partial_j^\alpha) &= [\nabla_{\partial_i^\alpha}, \nabla_{\partial_j^\alpha}] = [\partial_i^\alpha + A_i^\alpha, \partial_j^\alpha + A_j^\alpha] \\ &= (\partial_i^\alpha + A_i^\alpha)(\partial_j^\alpha + A_j^\alpha)\varphi - (\partial_j^\alpha + A_j^\alpha)(\partial_i^\alpha + A_i^\alpha) \\ &= \frac{\partial A_j^\alpha}{\partial x_i} + A_j^\alpha \partial_i^\alpha + A_i^\alpha \partial_j^\alpha + A_i^\alpha A_j^\alpha - \frac{\partial A_i^\alpha}{\partial x_j^\alpha} - A_i^\alpha \partial_j^\alpha - A_j^\alpha \partial_i^\alpha - A_j^\alpha A_i^\alpha \\ &= \frac{\partial A_j^\alpha}{\partial x_i} - \frac{\partial A_i^\alpha}{\partial x_j^\alpha} + [A_i^\alpha, A_j^\alpha] \\ &=: F_{ij}^\alpha \in \mathrm{gl}_k \mathbb{R}. \end{aligned}$$

We denote $F_\alpha := \sum_{i<j} F_{ij}^\alpha \mathrm{d}x_i^\alpha \mathrm{d}x_j^\alpha \in \Omega^2(U_\alpha, \mathrm{gl}_k \mathbb{R})$. To show the transition law we need another expression of curvature:

Lemma 4.8. $F_\alpha = \mathrm{d}A^\alpha + A^\alpha \wedge A^\alpha$.

Proof. $(\mathrm{d}A^\alpha + A^\alpha \wedge A^\alpha)(\partial_i^\alpha, \partial_j^\alpha) = (\mathrm{d}(\sum_i A_i^\alpha \mathrm{d}x_i^\alpha) + \sum_{i,j}(A_i^\alpha A_j^\alpha)\mathrm{d}x_i^\alpha \wedge \mathrm{d}x_j^\alpha)(\partial_i^\alpha, \partial_j^\alpha) = \frac{\partial A_j^\alpha}{\partial x_i} - \frac{\partial A_i^\alpha}{\partial x_j^\alpha} + [A_i^\alpha, A_j^\alpha] = F_{ij}^\alpha$. □

Remark 4.9. *This definition of curvature can be applied to any connections on a principal G-bundle.*

Since $A^\alpha = \mathrm{ad}_{g_{\alpha\beta}} A^\beta - \mathrm{d}g_{\alpha\beta} \cdot g_{\alpha\beta}^{-1}$, we have

$$\begin{aligned} F_\alpha =& \mathrm{d}\left(\mathrm{ad}_{g_{\alpha\beta}} A^\beta - \mathrm{d}g_{\alpha\beta} \cdot g_{\alpha\beta}^{-1}\right) + (\mathrm{ad}_{g_{\alpha\beta}} A^\beta - \mathrm{d}g_{\alpha\beta} \cdot g_{\alpha\beta}^{-1}) \wedge (\mathrm{ad}_{g_{\alpha\beta}} A^\beta - \mathrm{d}g_{\alpha\beta} \cdot g_{\alpha\beta}^{-1}) \\ =& g_{\alpha\beta}\mathrm{d}A^\beta g_{\alpha\beta}^{-1} + \mathrm{d}g_{\alpha\beta} \wedge A^\beta \cdot g_{\alpha\beta}^{-1} + g_{\alpha\beta} \cdot A^\beta \wedge \mathrm{d}g_{\alpha\beta}^{-1} - \mathrm{d}g_{\alpha\beta} \wedge g_{\alpha\beta} \cdot g_{\alpha\beta}^{-2} \\ & + \mathrm{ad}_{g_{\alpha\beta}} A^\beta \wedge A^\beta - \mathrm{ad}_{g_{\alpha\beta}} A^\beta \wedge \mathrm{d}g_{\alpha\beta} \cdot g_{\alpha\beta}^{-1} - \mathrm{ad}_{g_{\alpha\beta}} \mathrm{d}g_{\alpha\beta} \wedge A^\beta \cdot g_{\alpha\beta}^{-1} + \mathrm{d}g_{\alpha\beta} \wedge \mathrm{d}g_{\alpha\beta} \cdot g_{\alpha\beta}^{-2} \\ =& g_{\alpha\beta}\mathrm{d}A^\beta g_{\alpha\beta}^{-1} + \mathrm{ad}_{g_{\alpha\beta}} A^\beta \wedge A^\beta = \mathrm{ad}_{g_{\alpha\beta}} F_\beta, \end{aligned}$$

which completes the proof. □

5 Flat Connection, Parellel Transport, Holonomy

We can extend the connection from

$$\nabla : \Omega^0(M,E) \to \Omega^1(M,E)$$

to

$$\nabla : \Omega^p(M,E) \to \Omega^{p+1}(M,E)$$

by

$$\nabla : w \otimes s \mapsto \mathrm{d}w \otimes s + (-1)^p w \wedge \nabla s$$

where $w \in \Omega^p(M), s \in \Omega^0(M,E)$.

Recall that in a trivial bundle with trivial($A=0$) connection ($E = M \times \mathbb{R}^k \longrightarrow M, \nabla^{tri} = \mathrm{d}$), we have a sequence

$$0 \longrightarrow \Omega^0(M,\mathbb{R}^k) \xrightarrow{\mathrm{d}} \Omega^1(M,\mathbb{R}^k) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^k(M,\mathbb{R}^k) \longrightarrow 0.$$

It is called **de Rham complex** since $d \circ d = 0$. We also define **de Rham cohomology** by

$$H^p_{\mathrm{dR}}(M,\mathbb{R}^k) := \frac{\mathrm{Im}(\Omega^{p-1}(M,\mathbb{R}^k) \xrightarrow{\mathrm{d}} \Omega^p(M,\mathbb{R}^k))}{\mathrm{Ker}(\Omega^p(M,\mathbb{R}^k) \xrightarrow{\mathrm{d}} \Omega^{p+1}(M,\mathbb{R}^k))}.$$

It is not true that in a not necessary trivial bundle the sequence

$$0 \longrightarrow \Omega^0(M,\mathbb{R}^k) \xrightarrow{\nabla} \Omega^1(M,\mathbb{R}^k) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^k(M,\mathbb{R}^k) \longrightarrow 0.$$

is a complex; it depends on the curvature, i.e. whether ∇ is **flat**. In fact we have

Proposition 5.1. $F = \nabla \circ \nabla \in \Omega^2(M, \mathrm{End}\, E)$.

Therefore, the above sequence is a complex iff $F \equiv 0$. We now come to prove the proposition.

Proof. Under the trivialization $E|_{U_\alpha} \cong U_\alpha \times \mathbb{R}^k$ w.r.t. $\{\sigma_1^\alpha, \cdots, \sigma_k^\alpha\}$, we have

$$\begin{array}{ccccc} C^\infty(U_\alpha,\mathbb{R}^k) & \longrightarrow \Omega^1(U_\alpha,\mathbb{R}^k) & & \longrightarrow \Omega^2(U_\alpha,\mathbb{R}^k) \longrightarrow \cdots \\ \psi^\alpha = \begin{bmatrix} \psi_1^\alpha \\ \vdots \\ \psi_k^\alpha \end{bmatrix} & \mapsto (\mathrm{d} + A^\alpha)\psi^\alpha & & \mapsto \cdots . \end{array}$$

Hence,

$$
\begin{aligned}
&\nabla \circ \nabla(\{\sigma_1^\alpha, \cdots, \sigma_k^\alpha\}\psi^\alpha) \\
=&\nabla(\{\sigma_1^\alpha, \cdots, \sigma_k^\alpha\} \otimes (\mathrm{d} + A^\alpha)\psi^\alpha) \\
=&\{\sigma_1^\alpha, \cdots, \sigma_k^\alpha\} A^\alpha \wedge (\mathrm{d} + A^\alpha)\psi^\alpha + \{\sigma_1^\alpha, \cdots, \sigma_k^\alpha\} \otimes (\mathrm{d}A^\alpha\psi^\alpha - A^\alpha \wedge \mathrm{d}\psi^\alpha) \\
=&\{\sigma_1^\alpha, \cdots, \sigma_k^\alpha\} \otimes (\mathrm{d}A^\alpha + A^\alpha \wedge A^\alpha)\psi^\alpha \\
=&\{\sigma_1^\alpha, \cdots, \sigma_k^\alpha\} \otimes F^\alpha\psi^\alpha.
\end{aligned}
$$

□

Problem. Does flat connection always exist?

No. But the flat connection, if exists, can be induced from the universal cover.

Remark 5.2. *If there exists a flat connection, then all the Chern classes will vanish, since Chern classes are defined by curvature.*

We have another important property for the extended connection called Bianchi identity. By calculation, locally,

$$
\begin{aligned}
\mathrm{d}F^\alpha =&\mathrm{d}(\mathrm{d}A^\alpha + A^\alpha \wedge A^\alpha) \\
=&\mathrm{d}A^\alpha \wedge A^\alpha - A^\alpha \wedge \mathrm{d}A^\alpha \\
=&(F^\alpha - A^\alpha \wedge A^\alpha) \wedge A^\alpha - A^\alpha \wedge (F^\alpha - A^\alpha \wedge A^\alpha) \\
=&F^\alpha \wedge A^\alpha - A^\alpha \wedge F^\alpha \\
=&-[A^\alpha, F^\alpha],
\end{aligned}
$$

where the Lie bracket in the last line is wedge product on differential forms, i.e. the so-called *super commutator*. Therefore we have the **Bianchi identity** taking the form as

$$(\mathrm{d} + \mathrm{ad}\, A^\alpha)F^\alpha = 0,$$

where $\mathrm{ad}\, A^\alpha = [A^\alpha, -]$.

Compare it with $\Omega^2(M, \mathrm{End}\, E) \xrightarrow{\nabla} \Omega^3(M, \mathrm{End}\, E)$. Since

$$\mathrm{End}\, E = \mathrm{ad}\, \mathcal{F}r_{\mathrm{GL}_k} E = \mathcal{F}r_{\mathrm{GL}_k} E \times_{\mathrm{ad}} \mathrm{gl}_k \mathbb{R}$$

where

$$\mathrm{ad} : \mathrm{GL}(\mathbb{R}, k) \to \mathrm{GL}(\mathrm{gl}_k \mathbb{R}) : g \mapsto \mathrm{ad}_g = (A \mapsto gAg^{-1}),$$

we have induced connection 1-form(check Section 5) on $\operatorname{End} E$ as

$$\mathrm{d}|_I \operatorname{ad}(A^\alpha) \equiv \operatorname{ad} A^\alpha$$

where

$$\operatorname{ad} : \mathrm{gl}_k \mathbb{R} \to \mathrm{gl}(\mathrm{gl}_k \mathbb{R}) : A \mapsto \operatorname{ad} A = [A, -].$$

Be careful to the subtle difference on their notation between the two adjoint representation. We conclude that the Bianchi identity is amount to that

$$\nabla^{\operatorname{End} E} F \equiv 0.$$

Note that many books would simply say that $\nabla F = 0$, omitting the bundle that the connection is with respect to, which is $\operatorname{End} E$, rather than E.

Remark 5.3. *There is another deduction of the local connection 1-form for* $\operatorname{End} E$. *Choose a local frame* $\{s_1, \cdots, s_k\}$ *for* E *with its dual frame* $\{t^1, \cdots, t^k\}$. *Suppose* $\nabla^E s_i = \sum_k A_i^k s_k$, $\nabla^{E^*} t^j = \sum_k B_k^j t^k$ *The induced connection for* E^* *is given by*

$$0 = \mathrm{d}\left\langle s_i, t^j \right\rangle = \left\langle \nabla^E s_i, t^j \right\rangle + \left\langle s_i, \nabla^{E^*} t^j \right\rangle = A_i^j + B_j^i,$$

where $\langle -, - \rangle$ *is pairing. Thus* $B_j^i = -A_i^j$ *Noting that* $\operatorname{End} E = E \otimes E^*$, *we have*

$$\begin{aligned}
&\nabla^{E\otimes E^*} \sum_{i,j} a_j^i s_i \otimes t^j \\
=&\sum_{i,j} \mathrm{d}a_j^i \otimes s_i \otimes t^j + a_j^i (\nabla^E s_i) \otimes t^j + a_j^i s_i \otimes \nabla^{E^*} t^j \\
=&\sum_{i,j} (\mathrm{d}a_j^i + \sum_k a_j^k A_k^i - \sum_k a_k^i A_j^k) \otimes s_i \otimes t^j \\
=&\sum_{i,j} (\mathrm{d}a_j^i + (A\cdot a)_j^i - (a \cdot A)_j^i) \otimes s_i \otimes t^j \\
=&\sum_{i,j} (\mathrm{d} + \operatorname{ad} A) a_j^i \otimes s_i \otimes t^j.
\end{aligned}$$

We again derive that the local connection 1-form for $\operatorname{End} E$ *is* $\operatorname{ad} A^\alpha$, *where* A^α *is for* E.

Next we come to the third equivalent(again we would not show it) definition of connection, the parellel transport.

Given (E, ∇) over M and a smooth path $\gamma : [0, 1] \to M$ connecting $x_0, x_1 \in M$. With the pullback bundle and the pullback connection $(\gamma^* E, \gamma^* \nabla)$ via γ, the horizontal lifting of $\widetilde{\dot{\gamma}(t)}$ is a parellel transport, i.e. the solution $s(t)$ of the 1st order differential equation

$$\begin{cases} \nabla_{\dot\gamma(t)} s(t) = 0 \\ s(0) \in E_{x_0} \end{cases}.$$

The uniqueness of the solution is due to the compactness of the manifold.

If $\gamma(0) = \gamma(1)$, i.e. γ is a loop, we can define the **holonomy map** by $\mathrm{hol}_\nabla(\gamma) : E_{\gamma(0)} \to E_{\gamma(1)} : s(0) \mapsto s(1)$, or,

$$\mathrm{hol}_\nabla : \Omega_{x_0}(M) \to \mathrm{GL}(E_{x_0})$$

where Ω_{x_0} denotes the loop based at x_0.

Notice that $\mathrm{hol}_\nabla(\Omega_{x_0})$ has a group structure, i.e.

- $\mathrm{hol}_\nabla(\gamma) \circ \mathrm{hol}_\nabla(\gamma^{-1}) = \mathrm{Id}_{E_{x_0}}$,
- $\mathrm{hol}_\nabla(\gamma_1 \star \gamma_2) = \mathrm{hol}_\nabla(\gamma_1) \circ \mathrm{hol}_\nabla(\gamma_2)$.

Hence it give rise to the **holonomy group** $\mathcal{H}\mathrm{ol}_{x_0}(\nabla) = \mathrm{Im}\,\mathrm{hol}_\nabla < \mathrm{GL}(E_{x_0})$.

We have that

Theorem 5.4. *If ∇ is flat, then $\mathcal{H}\mathrm{ol}_{x_0}$ is a homotopy invariant.*

That is to say we have a holonomy representation for $\mathrm{GL}(E_{x_0})$.

$$\begin{tikzcd} \Omega_{x_0}(M) \arrow[r, "\mathrm{hol}_\nabla"] \arrow[d] & \mathrm{GL}(E_{x_0}) \\ \Omega_{x_0}(M)/\text{homotopy} \arrow[r, "\cong"'] & \pi_1(M, x_0) \arrow[u, "\text{holonomy representation}"'] \end{tikzcd}$$

6 Gauge Transformation, Gauge Group

An exercise. TBD.

Given a principal G-bundle P, the **gauge group** of P is defined by

$$\mathrm{Aut}(P) := \left\{ \begin{tikzcd} P \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & & P \arrow[dl, "\pi"] \\ & M & \end{tikzcd} \mid \psi \text{ is } G\text{-equivalent, i.e. } \psi(p \cdot g) = \psi(p) \cdot g, \forall p \in P, g \in G \right\}.$$

Given a gauge transform $\psi : P \to P$. For all $p \in P$, we can write $\psi(p \cdot g) = p \cdot g \cdot h(p \cdot g)$ as well as $\psi(p \cdot g) = \psi(p) \cdot g = p \cdot h(p) \cdot g$, which gives rise to a map $h : P \to G$ such that $h(p \cdot g) = g^{-1} \cdot h(p) \cdot g$.

We claim that h is *global* section of

$$\mathrm{Ad}\, P := P \times_{\mathrm{Ad}} G = \frac{P \times G}{(p, g) \sim (p \cdot g^{-1}, ghg^{-1})},$$

where $\mathrm{Ad} : G \to \mathrm{GL}(G) : g \mapsto (h \mapsto ghg^{-1})$.

Remark 6.1. *Notice that a global section does not exist on a principal bundle unless the bundle is trivial. However, such global section h does exist on the bundle $\mathrm{Ad}\, P$; let $h(p) \equiv \mathrm{id}$.*

Note that a global section of $\mathrm{Ad}\, P$ is given by $\{s_\alpha : U_\alpha \to G\}_\alpha$ satisfying $s_\alpha = \mathrm{Ad}_{g_{\alpha\beta}} s_\beta$ on $U_\alpha \cap U_\beta$. (This is because on $U_\alpha \cap U_\beta$ with corresponding local section $\sigma_\alpha, \sigma_\beta$, we have $[(\sigma_\beta, s_\beta)] = [(\sigma_\alpha g_{\alpha\beta}, s_\beta)] = [(\sigma_\beta, \mathrm{Ad}_{g_{\alpha\beta}} s_\beta)]$.) We set $h_\alpha : U_\alpha \to G : x \mapsto h(\sigma_\alpha(x))$. Then for $x \in U_\alpha \cap U_\beta$, we have $h_\beta(x) = h(\sigma_\beta(x)) = h(\sigma_\alpha(x) g_{\alpha\beta}(x)) = \mathrm{Ad}_{g^{-1}_{\alpha\beta}(x)} h(\sigma_\alpha(x)) = \mathrm{Ad}_{g^{-1}_{\alpha\beta}(x)} h_\alpha(x)$, which proves our claim.

To sum up, the gauge (transformation) group of P is given by

$$\begin{aligned} \mathcal{G}_P &= \mathrm{Aut}(P) = \Gamma(M, \mathrm{Ad}\, P) \\ &= \{h : P \to G : h(p \cdot g) = g^{-1} \cdot h(p) \cdot g\} \\ &= \{\{h_\alpha : U_\alpha \to G\}_\alpha : h_\alpha(x) = \mathrm{Ad}_{g_{\alpha\beta}} h_\beta \text{ on } U_\alpha \cap U_\beta\}. \end{aligned}$$

Remark 6.2 (TBD.). *The Lie algebra of $\mathrm{Aut}(P)$ is given by $\Omega^0(M, \mathrm{ad}\, P)$.*

Let gauge group act on the space of connection by

$$\mathcal{G}_P \times \mathcal{A}_P \to \mathcal{A}_P : (h, A) \mapsto h \cdot A.$$

Locally we have

$$\mathcal{G}_P = \{\{h_\alpha : U_\alpha \to G\}_\alpha : h_\alpha(x) = \mathrm{Ad}_{g_{\alpha\beta}} h_\beta \text{ on } U_\alpha \cap U_\beta\}$$

as well as

$$\mathcal{A}_P = \{\{A_\alpha \in \Omega^1(U_\alpha, \mathrm{gl}_k \mathbb{R})\}_\alpha : A_\alpha = \mathrm{ad}_{g_{\alpha\beta}} A_\beta - \mathrm{d}g_{\alpha\beta} \cdot g_{\alpha\beta}^{-1} \text{ on } U_\alpha \cap U_\beta\}.$$

Therefore we can define $h \cdot A$ locally by

$$(h \cdot A)_\alpha := \mathrm{ad}_{h_\alpha} A_\alpha - \mathrm{d}h_\alpha \cdot h_\alpha^{-1} \tag{*}$$

which is in $\mathcal{A}_P$ since

$$\begin{aligned}
(h \cdot A)_\alpha &= \mathrm{ad}_{h_\alpha} A_\alpha - \mathrm{d}h_{\alpha\beta} \cdot h_{\alpha\beta}^{-1} \\
&= g_{\alpha\beta} h_\beta g_{\alpha\beta}^{-1} (\mathrm{ad}_{g_{\alpha\beta}} A_\beta - \mathrm{d}g_{\alpha\beta} \cdot g_{\alpha\beta}^{-1}) g_{\alpha\beta} h_\beta^{-1} g_{\alpha\beta}^{-1} - \mathrm{d}(g_{\alpha\beta} h_\beta g_{\alpha\beta}^{-1}) \cdot (g_{\alpha\beta} h_\beta g_{\alpha\beta}^{-1})^{-1} \\
&= g_{\alpha\beta} h_\beta A_\beta h_\beta^{-1} g_{\alpha\beta}^{-1} - g_{\alpha\beta} h_\beta g_{\alpha\beta}^{-1} \mathrm{d}g_{\alpha\beta} h_\beta^{-1} g_{\alpha\beta}^{-1} - \mathrm{d}g_{\alpha\beta} \cdot g_{\alpha\beta}^{-1} - g_{\alpha\beta} \mathrm{d}h_\beta \cdot h_\beta^{-1} g_{\alpha\beta}^{-1} + g_{\alpha\beta} h_\beta g_{\alpha\beta}^{-1} \mathrm{d}g_{\alpha\beta} h_\beta^{-1} g_{\alpha\beta}^{-1} \\
&= \mathrm{ad}_{g_{\alpha\beta}} (\mathrm{ad}_{h_\beta} A_\beta - \mathrm{d}h_\beta \cdot h_\beta^{-1}) - \mathrm{d}g_{\alpha\beta} \cdot g_{\alpha\beta}^{-1} \\
&= \mathrm{ad}_{g_{\alpha\beta}} (h \cdot A)_\beta - \mathrm{d}g_{\alpha\beta} \cdot g_{\alpha\beta}^{-1}.
\end{aligned}$$

In the third line we have used the fact that $\mathrm{d}(g^{-1}) = -g^{-1} \cdot \mathrm{d}g \cdot g^{-1}$, since $0 = \mathrm{d}(g \cdot g^{-1}) = \mathrm{d}g \cdot g^{-1} + g \cdot \mathrm{d}(g^{-1})$.

Remark 6.3. *Note that if G is a matrix Lie group, then $\mathrm{d}h_\alpha$ is nothing other than taking differential on each entry. If it is an abstract Lie group, then $\mathrm{d}h_\alpha \cdot h_\alpha^{-1}$ is the pullback of the Mawer-Cartan 1-form on G, i.e. $h_\alpha^*(\mathrm{d}g \cdot g^{-1})$.*

We also have a vector bundle version of gauge group. Given a vector bundle E, we define

$$\mathcal{G}_E \equiv \mathrm{Aut}(E) := \left\{ \begin{tikzcd} E \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & & E \arrow[dl, "\pi"] \\ & M & \end{tikzcd} \mid \psi \text{ is a linear isomorphism along each fiber.} \right\}$$

Similarily as above we have that

$$\mathcal{G}_E = \Gamma(M, \mathcal{F}r_{\mathrm{GL}(k)} E \times_{\mathrm{Ad}} \mathrm{GL}(k)).$$

We define the gauge group action on the space of connection by the commutative diagram as below:

$$\begin{tikzcd} \mathfrak{X}(M) \times \Gamma(E) \arrow[r, "\nabla"] \arrow[d, "\mathrm{id} \times \psi"'] & \Gamma(E) \arrow[d, "\psi"] \\ \mathfrak{X}(M) \times \Gamma(E) \arrow[r, "\nabla^\psi"'] & \Gamma(E) \end{tikzcd}$$

i.e.

$$\nabla^{\psi} = \psi \circ \nabla \circ (\mathrm{id} \times \psi)^{-1},$$

whose local expression also satisfies $(*)$ since

$$(A^{\psi})_{\alpha} = \psi_{\alpha}(A_{\alpha}\psi_{\alpha}^{-1} + \mathrm{d}(\psi_{\alpha}^{-1})) = \psi_{\alpha}A_{\alpha}\psi^{-1} - \mathrm{d}\psi_{\alpha} \cdot \psi_{\alpha}^{-1},$$

where the notation is self-evident.

7 Stablizer of Gauge Transformation, Hilbert Norm

Recall that a Lie group G **acts on** a smooth manifold M is given by

$$G \times M \to M : (g, m) \mapsto g \cdot m.$$

In other words, if we fix $g \in G$, we will derive a diffeomorphism $m \mapsto g \cdot m$, i.e.

$$G \to \mathrm{Diff}(M)$$

is a Lie group homomorphism. The Lie algebra of $\mathrm{Diff}(M)$ is given by $\Gamma(TM)$. The **stablizer** of $x \in M$ is defined to be

$$G_x := \{g \in G : g \cdot x = x\} < G,$$

obviously a Lie subgroup of G. The Lie algebra of G_x is therefore

$$\mathfrak{g}_x \equiv \mathrm{Lie}(G_x) := \{\xi \in \mathfrak{g} : \tilde{\xi}_x = 0\} < \mathfrak{g},$$

where $\tilde{\bullet} : \mathfrak{g} \to \Gamma(TM) : \xi \mapsto (p \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\exp(t\xi) \cdot p))$. A **free** action means that $\forall x \in M, G_x = \{0\}$, while a **trivial** action means that $\forall x \in M, G_x = G$. If the G-action is free, the quotient space M/G is a smooth manifold. We have a Slice Theorem which claims that

Theorem 7.1 (Slice Theorem). $T_xM = T_x(G \cdot x) \oplus (T_x(G \cdot x))^{\perp}$, *where the direct sum is with respect to some metric.*

Under the exponential map, an open ball in $(T_x(G \cdot x))^{\perp}$ becomes a **slice**.

Gauge theory can be stated as to study and solve certain PDE for a connection A up to gauge equivalence, i.e. our aimed space is

$$\mathcal{A}(P)/\mathcal{G}(P).$$

We now introduce the **Hilbert norm**. Let M^n be compact and oriented. For $C^{\infty}(M, \mathbb{R})$, we equip it a inner product as

$$\langle f, g\rangle_{L^2} := \int_M |f||g| \, \mathrm{dVol}_M$$

where dVol_M is a nonzero section on the determinant bundle $\det(T^*M) \equiv \wedge^n T^*M \longrightarrow M$. The induced metric is hence

$$\|f - g\|_{L^2}^2 := \langle f - g, f - g\rangle_{L^2}.$$

By completion, we derive the L^2-complete space $L^2(M,\mathbb{R})$. We can define $L^2(M,\mathbb{R}^k)$ similarily. Since $\Omega^0(M,E)$ is locally $\Omega^0(U_\alpha,\mathbb{R}^k)$, we also have $\Omega^0_{L^2}(M,E)$ being the L^2-completion of $\Omega^0(M,E)$ via local completions. Notice that $\mathrm{Ad}\,P = P\times_{\mathrm{Ad}} G$ as well as that $G\hookrightarrow\mathbb{R}^N$ for some large enough N, we can also define $\Omega^0_{L^2}(M,\mathrm{Ad}\,P)$.

To define $\Omega^k_{L^2}(M,E)$, we should first introduce Hodge star operator $*:\Omega^k\to\Omega^{n-k}$ defined in basis by

$$*(\mathrm{d}x_{i_1}\wedge\cdots\wedge\mathrm{d}x_{i_k}) = \varepsilon_{i_1,\cdots i_n}\mathrm{d}x_{i_{k+1}}\wedge\cdots\wedge x_{i_n},$$

where $\varepsilon_{i_1,\cdots,i_n} = \mathrm{sgn}\begin{pmatrix}1 & \cdots & n\\ i_1 & \cdots & i_n\end{pmatrix}$. For $\alpha,\beta\in\Omega^k(M,\mathbb{R})$, the inner product is set to be

$$\langle\alpha,\beta\rangle_{L^2} := \int_M \alpha\wedge*\beta.$$

Hence by completion we derive $\Omega^k_{L^2}(M,\mathbb{R})$. For vector bundle, via the standard process of taking a partition of unity $\{\rho_\alpha\}$, we define

$$\langle\alpha,\beta\rangle_{L^2} = \sum_\alpha\int_{U_\alpha}\rho_\alpha\alpha|_{U_\alpha}\wedge*\beta|_{U_\alpha}$$

for $\alpha,\beta\in\Omega^k(M,E)$. By completion again we get $\Omega^k_{L^2}(M,E)$.

The L^2-norm can be extended to L^2_k-norm, where we define

$$\|f\|^2_{L^2_k} := \int_M |f|^2+|\nabla f|^2+\cdots+|\nabla^k f|^2\,\mathrm{dVol}_M < +\infty.$$

By completion *mutatis mutandis*, we have $L^2_k(M,\mathbb{R})$ and moreover $\Omega^k_{L^2_k}(M,E)$. Recall that

Theorem 7.2 (Sobolev Embedding)**.** *If $k-\frac{n}{2}>0$, then there exists a continuous embedding $L^2_k\hookrightarrow C^0$.*

We have some results that

Theorem 7.3 ([Freed-Uhlenbecks)**.** *] Assume that $k-\frac{n}{2}>0$.*

1. *$\mathcal{G}_{L^2_k}(P)$ is a Hilbert Lie group with its Lie algebra $\Omega^0_{L^2_k}(M,\mathrm{ad}\,P)$.*
2. *$\mathcal{G}_{L^2_k}(P)\times\mathcal{A}_{L^2_{k-1}}(P)\to\mathcal{A}_{L^2_{k-1}}(P)$ is smooth, where $\mathcal{A}_{L^2_{k-1}}$ is an affine Hilbert space.*
3. *The curvature operator $F:\mathcal{A}_{L^2_{k-1}}(P)\to\Omega^2_{L^2_{k-2}}(M,\mathrm{ad}\,P)$ is smooth.*

Here smooth is in the following sense. Suppose there exists a map $f:\mathcal{M}\to\mathcal{N}$ between two Hilbert manifold $\mathcal{M},\mathcal{N}$. If $\mathrm{D}^k f: T_x\mathcal{M}\times\cdots\times T_x\mathcal{M}\to T_{f(x)}\mathcal{N}$ is continuous, then we call f is of C^k-class. Therefore by smooth we mean that the map is of C^∞-class.

For all $\xi \in \Omega^0_{L^2_k}(M, \mathrm{ad}\, P) = \mathrm{Lie}\, \mathcal{G}_{L^2_k}(P)$, it generates a vector field on $\mathcal{A}_{L^2_{k-1}}$ by

$$\tilde{\xi}_A = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\exp(t\xi) \cdot A) \,.$$

Locally,

$$\begin{aligned}(\tilde{\xi}_A)^\alpha &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Big(\mathrm{Ad}_{\exp(\xi^\alpha)} A^\alpha - \mathrm{d}\exp(t\xi^\alpha)\exp(-t\xi^\alpha)\Big) \\ &= \cdots \\ &= -\,(\mathrm{d} + [A^\alpha, -])\xi^\alpha,\end{aligned}$$

i.e.

$$\tilde{\xi}_A = -\nabla_A^{\mathrm{ad}\, P} \xi.$$

Later we will omit the superscript $\mathrm{ad}\, P$ for brevity. Recall that the stablizer is given by $\mathcal{G}_A = \{h \in \mathcal{G}(P) : h \cdot A = A\}$ and its Lie algebra by $\mathfrak{g}_A = \{\xi \in \Omega^0_{L^2_k}(M, \mathrm{ad}\, P) : \tilde{\xi}_A = 0\} = \mathrm{Ker}\, \nabla_A$. Recall also that $\Omega^1_{L^2_{k-1}}(M, \mathrm{ad}\, P) = T_A\, \mathcal{A}_{L^2_{k-1}}(P)$. Therefore we have

$$T_A(\mathcal{G} \cdot A) = \mathrm{Im}\, \nabla_A$$

and a $\mathcal{G}_A$-invariant slice given by

$$A + (\mathrm{Im}\, \nabla_A)^\perp \equiv A + \mathrm{Ker}\, \nabla_A^*,$$

where $\nabla_A^* : \Omega^1_{L^2_{k-1}}(M, \mathrm{ad}\, P) \to \Omega^0_{L^2_k}(M, \mathrm{ad}\, P)$ is the dual operator of $\nabla_A : \Omega^0_{L^2_k}(M, \mathrm{ad}\, P) \to \Omega^1_{L^2_{k-1}}(M, \mathrm{ad}\, P)$ w.r.t. some inner products equipped on the two spaces.

8 Reducible & Irreducible Connection, Quotient Space of $\mathcal{A}_{L^2_{k-1}}(P)/\mathcal{G}_{L^2_k}(P)$

Recall that the stablizer of a given connection A and its Lie algebra is given by

$$\mathcal{G}_A = \{g \in \mathcal{G} : g \cdot A = A\},$$

$$\mathfrak{g}_A = \{\xi \in \Omega^0(M, \operatorname{ad} P) : \nabla_A \xi = 0\}.$$

Consider the following examples.

Example 8.1. *A trivial bundle $M \times G$ together with the trivial connection $A = \theta$. Since $\nabla_\theta \xi = \mathrm{d}\xi = 0$ implies that ξ is constant in $\Omega^0(M, \mathfrak{g})$, we have that $\mathfrak{g}_\theta = \mathfrak{g}$, then $\mathcal{G}_\theta = \exp \mathfrak{g}_\theta = G$, the group of **constant gauge transformations**.*

Example 8.2. *A principal $\mathrm{U}(1)$-bundle P. Since $\mathrm{U}(1)$ is abelian, we yield that $\operatorname{ad} P, \operatorname{Ad} P$ are trivial. Note that the Lie algebra of $\mathrm{U}(1) = \{\mathrm{e}^{\mathrm{i}\,\theta}\}$ is $\mathrm{i}\,\mathbb{R}$. We have $\mathcal{A} = A_0 + \Omega^1(M, \mathrm{i}\,\mathbb{R}), \mathcal{G} = C^\infty(M, \mathrm{U}(1)), \mathfrak{g} = C^\infty(M, \mathrm{i}\,\mathbb{R})$, leading to $\mathfrak{g}_A = \{\xi \in \mathfrak{g} : \mathrm{d}\xi + [A, \xi] = \mathrm{d}\xi = 0\} = \mathrm{i}\,\mathbb{R}$ as well as $\mathcal{G}_A = \exp \mathfrak{g}_A = \mathrm{U}(1)$ for any connection A.*

Example 8.3. *A principal $\mathrm{SU}(2)$-bundle P, where $\mathrm{SU}(2) = \{g \in \mathrm{GL}(2, \mathbb{C}) : \bar{g}^T g = I_{2\times 2}, \det g = 1\} \cong \mathbb{S}^3$. If all transition functions of P is taken value as $g_{\alpha\beta} = \begin{bmatrix} \mathrm{e}^{\mathrm{i}\,\theta_{\alpha\beta}} & \\ & \mathrm{e}^{-\mathrm{i}\,\theta_{\alpha\beta}} \end{bmatrix}$, then the associated rank 2 complex vector bundle can be reduced to a principal $\mathrm{U}(1)$-bundle*

$$P \times_{\rho_0} \mathbb{C}^2 \cong L \oplus L^*,$$

where $\rho_0 : G \to \operatorname{Aut}(\mathbb{C}^2)$ is the canonical representation.

*If there is a connection taking the form as $A = \left\{ A^\alpha = \begin{bmatrix} a^\alpha & \\ & -a^\alpha \end{bmatrix}, a^\alpha \in \Omega^1(U_\alpha, \mathrm{i}\,\mathbb{R}) \right\}$, then $\mathfrak{g}_A = \{\xi : \mathrm{d}\xi + [A, \xi] = \mathrm{d}\xi = 0\} \cong \mathrm{i}\,\mathbb{R}$ as well as $\mathcal{G}_A = \exp \mathfrak{g}_A \cong \mathrm{U}(1)$. Such connection that is called **reducible**.*

Note that $\mathbb{Z}_2 < \mathcal{G}$.

Definition 8.4. *A connection A is called **irreducible** if $\mathcal{G}_A = \mathbb{Z}_2$.*

Let $\mathcal{A}^*$ denote the set of all the irreducible connections. Since $\mathcal{G}/\mathbb{Z}^2$ acts on A^* freely, $\mathcal{A}^* / \mathcal{G}$ is automatically a smooth Hilbert manifold, with tangent space at A given by $\operatorname{Ker}(\nabla_A^* : \Omega^1(M, \operatorname{ad} P) \to \Omega^0(M, \operatorname{ad} P))$.

If A_0 is irreducible, $\mathcal{G}_{A_0} = \mathrm{U}(1)$, we have

$$T_{A_0}\mathcal{A} = T_{A_0}(\mathcal{G}\cdot A_0) \oplus \mathrm{Ker}(\nabla_A^*).$$

Since $T_{A_0}\mathcal{A}$ is U(1)-invariant, the latter direct sum component can be decomposed as

$$\mathrm{Ker}(\nabla_A^*) = \mathcal{H}_{\mathbb{R}} \oplus \mathcal{H}_{\mathbb{C}},$$

where $\mathcal{H}$ is real Hilbert space. The local structure of $\mathcal{A}/\mathcal{G}$ at a reducible connection $[A_0]$ is given by

$$\mathcal{H}_{\mathbb{R}} \oplus (\mathcal{H}_{\mathbb{C}} / \mathrm{U}(1)).$$

Gauge theory studies some elliptic PDE mostly from classical equation of motions. For example,

- $F_A \equiv 0$ for flat connection from 2D/3D Yang-Mills Theory;
- Yang-Mills function for 4D Yang-Mills Theory. Given a principal SU(2)-bundle P over M^4, the Yang-Mills functional is given by

$$\mathrm{YM} : \mathcal{A} \to \mathbb{R} : A \mapsto \|F_A\|^2_{L^2} = -\int_M \mathrm{Tr}(F_A \wedge *F_A).$$

We would like to consider the critical points of such map

$$\mathrm{crit(YM)} := \left\{ A \mid \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\mathrm{YM}(A_t)) = 0 \text{ for any family of connections passing through } A \right\}.$$

In fact, we have

$$\mathrm{crit(YM)} = \{A : \nabla_A^* F_A = 0\}.$$

9 3D Gauge Theory (Chern-Simons Theory)

Given a 3-dim closed Riemannian manifold Y^3 with $P \longrightarrow Y$ a G-bundle over it. We suppose $G = \mathrm{SO}(3)$ or $\mathrm{SU}(2)$ for simplicity in the following context.

Proposition 9.1. *Any* $\mathrm{SU}(2)$*-principal bundle over* Y *is trivial.*

Proof sketch. Firstly we applied Heegaard splitting $Y = H_1 \cup_{\Sigma_g} H_2$, where H_i's are solid g-tori. We can explain as follows: we choose a Morse-Smale function f and consider its critical points. We choose a suitable k such that $f^{-1}(k)$ split the manifold into 2 components $\Pi_1 := \{x : f(x) \le k\}, \Pi_2 := \{x : f(x) \ge k\}$ with Π_1 only contains those critical point of index 0, 1 and Π_2 of index 2, 3. Since $f, -f$ are both Morse-Smale function, it is easy to know that the number of critical points indexed 1 is equal to the number of those indexed 2, denoted by g. Π_1 is clearly homeomorphic to a 3-ball with g solid handles($D^1 \times D^2$) attached, as well as Π_2.[DFN]

Then note that the bundle $P_1 \longrightarrow H_1, P_2 \longrightarrow H_2$ are both trivializable. This is because H_i's are both solid g-tori, which can be decomposed into hemispheres and solid pants that are all contractible, with contractible boundaries of disks.

Lastly, since all mappings $g : \Sigma_g \to \mathrm{SU}(2)$ are null-homotopic, as well as the homeomorphism class of the bundle $P_1 \cup_g P_2$ is only dependent on homotopic class of g (lemma 1.4.6 of [A]), the proof is complete. □

Therefore $\mathrm{ad}\, P, \mathrm{Ad}\, P$ are all trivial.

Let $\mathcal{A}(P)$ be the space of SU(2)-connection on $Y \times \mathrm{SU}(2)$ and $\mathcal{G}(P)$ the completion of the gauge group $L^2_k(Y, \mathrm{SU}(2))$. To make $L^2_k \hookrightarrow C^0$, we let $k - \frac{n}{2} > 0$. Since $n = 3$, we take $k = 2$.

The quotient of the space of connections by gauge transformation has a decomposition

$$\mathcal{B}(P) \equiv \mathcal{A}(P)/\mathcal{G}(P) = \{[\theta]\} \sqcup \mathcal{B}^{\mathrm{red}}(P) \sqcup \mathcal{B}^*(P),$$

where θ is the trivial connection, $\mathcal{B}^*(P)$ the space of irreducible connections, which is a ∞-dim smooth Hilbert manifold. $\mathcal{B}(P)$ is Hausdorff([1]).

We define Chern-Simons function as

$$\mathrm{CS} : \mathcal{A}(P) = \theta + \Omega^1(Y, \mathfrak{su}(2)) \to \mathbb{R} : A \mapsto CS(A),$$

by abusing the notation, we assume $A \in \Omega^1(Y, \mathfrak{su}(2))$. There are two fashions of definitions, one is

- $$\mathrm{CS}(A) := \frac{1}{8\pi^2}\int_Y \mathrm{Tr}(A\wedge \mathrm{d}A + \frac{2}{3}A\wedge A\wedge A);$$

  and the other is

- via the "relative" 2nd Chern class of a principal SU(2)-bundle over a 4-manifold X with boundary Y.

  Firstly we suppose $\partial X^4 = \varnothing$. The 2nd Chern class is $c_2(P_X) = [\frac{1}{8\pi^2}\mathrm{Tr}(F_{\mathbb{A}}\wedge F_{\mathbb{A}})] \in H^4(X,\mathbb{Z})$, where $\mathbb{A}$ is a SU(2)-connection on P_X. The 2nd Chern class is then

  $$c_2 = \int_X \frac{1}{8\pi^2}\mathrm{Tr}(F_{\mathbb{A}}\wedge F_{\mathbb{A}}) \in \mathbb{Z}.$$

  Next we suppose $\partial X^4 = Y^3$, therefore $\partial P_X = P_Y$. Note that P_Y is trivial, we can always choose a connection $\mathbb{A}$ on P_X such that $\mathbb{A}|_{P_y} = A$. We can define

  $$\mathrm{CS}(A) = \int_X \mathrm{Tr}(F_{\mathbb{A}}\wedge F_{\mathbb{A}}) \pmod{\mathbb{Z}}$$

  which is independent of the choice of $\mathbb{A}$, since we can choose two manifold-with-boundary X_1, X_2 such that $\partial X_1 = \partial X_2 = Y$, and then combine them into a close manifold $X = X_1 \cup_Y X_2$ as well as the connections on both bundle into one connection via the indentity glueing function. From the first case we know that 2nd Chern number of every close manifold is an integer, comparing to the fact that the difference $\int_{X_1}\mathrm{Tr}(F_{\mathbb{A}_1}\wedge F_{\mathbb{A}_1}) - \int_{X_2}\mathrm{Tr}(F_{\mathbb{A}_2}\wedge F_{\mathbb{A}_2})$ gives rise to a 2nd Chern number of a close manifold, we derive that $\mathrm{CS}(-)$ is determined up to an integer.

We have

Proposition 9.2. *1. The two definition above are equivalence.*

2. For each gauge transform $g \in \mathrm{Aut}(P_Y) = L^2_2(Y, \mathrm{SU}(2))$, we have

$$\mathrm{CS}(g\cdot A) - \mathrm{CS}(A) \in \mathbb{Z}$$

.

Proof. The first statement is came from Stokes's formula and calculations. The second is due to that

$$\mathrm{CS}(g\cdot A) - \mathrm{CS}(A) = \frac{1}{24\pi^2}\int_Y \mathrm{Tr}(g^{-1}\mathrm{d}g)^3 = \deg(g: Y^3 \to \mathrm{SU}(2)) \in \mathbb{Z},$$

since $\deg g = \int_Y g^*\Theta$ and $[\Theta] = [\frac{1}{24\pi^2}(g^{-1}\mathrm{d}g)^3] \in H^3(\mathrm{SU}(2),\mathbb{Z}) \cong \mathbb{Z}$, where the isomorphism is due to dimension of Y, under which isomorphism $[\Theta] \mapsto 1$. ∎

So far we have known that

$$\mathcal{B} = \mathcal{A}/\mathcal{G} \xrightarrow{\text{CS}} \mathbb{R}/\mathbb{Z}.$$

Recall that in Morse theory [M, S], for a Morse function f, we have

$$\chi(M) = \sum_{x \in (\nabla f)^{-1}(0)} (-1)^{\operatorname{ind} x}$$

where χ is Euler's number. And if f further satisfies Smale's condition, we have a chain complex

$$C_*(M, f) := \bigoplus_{x \text{ critical point}} \mathbb{Z}\langle x \rangle$$

graded by the index of critical points and a boundary operator

$$\partial : \langle x \rangle \mapsto \sum_{\operatorname{ind} x = \operatorname{ind} y + 1} n_{xy} \langle y \rangle$$

where $n_{xy} := \#(S_x^u \cap S_y^s/\mathbb{R})$ is the counting number of the intersection of the stable and the unstable manifold modulo the time. So far we have defined a homology theory $H_*(M, f)$ which is surprisingly not dependent on the choice of f. Moreover we can find out that

$$H_*(M, f) \cong H_*(M; \mathbb{Z}).$$

Note that we have derived some topological information and have recovered the homology group from a Morse-Smale function. We may wonder

Question. Can we do something similar for the Chern-Simons function on $\mathcal{B}$?

Yes. But with some conditions.

Firstly we need to study what the critical points are.

Proposition 9.3. $(\nabla \operatorname{CS})^{-1}(0)$ *consists of flat connections.*

The gradient is with respect to the L^2-inner product of $T_A \mathcal{A} = \Omega^1_{L^2}(Y, \mathfrak{su}(2))$. Namely $\langle \eta_1, \eta_2 \rangle = -\int_Y \operatorname{Tr}(\eta_1 \wedge *\eta_2)$.

Proof. For every $A \in (\nabla \operatorname{CS})^{-1}(0)$, choose a smooth path $A + t\eta$ on $\mathcal{A}$ through A for all $\eta \in \Omega^1(Y, \mathfrak{su}(2))$. At critical point we have

$$\begin{aligned}
0 &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\operatorname{CS}(A + t\eta)) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\frac{1}{8\pi^2} \int_Y \operatorname{Tr}\left((A + t\eta) \wedge \mathrm{d}(A + t\eta) + \frac{2}{3}(A + t\eta)^3 \right) \right) \\
&= \frac{1}{8\pi^2} \int_Y \operatorname{Tr}(\eta \wedge \mathrm{d}A + A \wedge \mathrm{d}\eta + 2\eta \wedge A \wedge A) \\
&= \frac{1}{4\pi^2} \int_Y \operatorname{Tr}(\eta \wedge F_A) \\
&= -\frac{1}{4\pi^2} \langle \eta, *F_A \rangle .
\end{aligned}$$

where the third line is due to Stokes' Formula. Therefore we have

$$\nabla \operatorname{CS}(A) = -\frac{1}{4\pi^2} * F_A \in \Omega^1(Y, \mathfrak{su}(2)).$$

□

Remark 9.4. *Hence we know that* $\nabla \operatorname{CS}(A) = - * F_A$ *is a section of tangent bundle* $T\mathcal{A}$*. If we take irreducible connections modulo gauge transform, we derive a section* $[A, - * F_A]$ *of tangent bundle* $T\mathcal{B}^* \longrightarrow \mathcal{B}^*$*.* $\mathcal{B}^*$ *is a smooth Hilbert manifold.* $T\mathcal{B}^*$ *is a subbundle of* $\mathcal{A}^* \times_{\mathcal{G}(P)} \Omega^1(X, \mathfrak{su}(2))$*, since we have*

$$T_{[A]}\mathcal{B}^* = \operatorname{Ker}(\mathrm{d}_A^* : \Omega^1(X, \mathfrak{su}(2)) \to \Omega^0(X, \mathfrak{su}(2)))$$

from $\mathrm{d}_A^*(*F_A) = - * d_A * (*F_A) = - * \mathrm{d}_A F_A = 0$*, where the last equality is Bianchi's identity. We call* $\mathcal{B}^* \supset (\nabla \operatorname{CS})^{-1}(0)$ ***the moduli space of flat connections****.*

Theorem 9.5 (Taubes)**.** *For any homological 3-sphere* Y*, i.e.* $H_i(Y; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 0, 3 \\ 0, & i = 1, 2 \end{cases}$*, with relative index defined on* $(\nabla \operatorname{CS})^{-1}(0)$ *given by*

$$\operatorname{ind}(A_1, A_2) := \textit{Fredholm index of } \partial_t + \operatorname{Hess}(\operatorname{CS})(A_t),$$

for non-trivial A_1, A_2*, then*

$$\#\textit{ of irreducible flat connection on } P = Y \times \operatorname{SU}(2) \textit{ modulo gauge transform} = 2 \cdot \textit{Casson invariant of } Y.$$

Remark 9.6. *The homological 3-sphere force the reducible connections to disappear, i.e.* $(\nabla \operatorname{CS})^{-1}(0) = \{\theta\} \sqcup \{[A] : F_A = 0, A : \textit{irreducible}\}$*.*

The anologue of Morse homology is Floer instanton homology of (Y, P) where Y is homological 3-sphere.

Summary. So far for a homological 3-sphere, we have recovered the classical casson invariants. We can also define other invariants via holonomy over some knot.

10 2D Yang-Mills Theory

We consider the principal bundle over a Riemannian surface $P \longrightarrow \Sigma_g^2$, where the base space is topologically a $\#_g T^2$ and is with a complex structure J. Recall that by complex structure we mean an automorphism on $T\Sigma^2$, such that $J^2 = -\operatorname{id}_{T\Sigma}$, i.e. in local coordinates $J(\partial_x) = \partial_y, J(\partial_y) = -\partial_x$. $T\Sigma$ is itself a complex bundle since $\mathbb{R}^2 \cong \mathbb{C}$. We have an eigenspace decomposition for the complexification space $T\Sigma \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}\Sigma \oplus T^{0,1}\Sigma$, whose generators are $\partial_z, \partial_{\bar{z}}$, respectively, where $z = x + \mathrm{i}\, y$, since $J(\partial_z) = \mathrm{i}\,\partial_z, J(\partial_{\bar{z}}) = -\mathrm{i}\,\partial_{\bar{z}}$.

There is also a complex structure on the space of connections $\mathcal{A}(P)$; the Hodge star operator on Σ gives rise to the complex structure we need, namely $* : \Omega^1(\Sigma, \operatorname{ad} P) \to \Omega^1(\Sigma, \operatorname{ad} P)$ satisfying $*^2 = \operatorname{id}$ and in coordinate $*\mathrm{d}x = \mathrm{d}y, *\mathrm{d}y = -\mathrm{d}x$. Note also that $T\,\mathcal{A}(P) = \mathcal{A}(P) \times \Omega^1_{L^2_1}(\Sigma, \operatorname{ad} P)$ is a Hilbert bundle. Therefore we derive

Lemma 10.1. *$T\,\mathcal{A}(P)$ is a complex Hilbert bundle.*

Remark 10.2. *An element $\xi \in \Omega^1(\Sigma, \operatorname{ad} P)$ can be expressed locally on an open set U as $\xi = \sum_{a=1}^d \xi_a T^a$ for $\xi_a \in \Omega^1(U)$ and $\{T^a\}$ is the basis of $\operatorname{ad} P$.*

Remark 10.3. *We also have $T\,\mathcal{A} \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}\,\mathcal{A} \oplus T^{0,1}\,\mathcal{A}$.*

Atiyah and Bott proved the following results in [AB82].

Theorem 10.4.
1. *$\mathcal{A}(P)$ has a symplectic structure and a compatible complex structure $J_{\mathcal{A}} = *$.*
2. *The gauge group $\mathcal{G}(P)$ action on $\mathcal{A}(P)$ is Hamiltonion with moment map $\mu : \mathcal{A}(P) \to (\operatorname{Lie}\mathcal{G}(P))^*$ under the identification $(\operatorname{Lie}\mathcal{G}(P) = \Omega^0(\Sigma, \operatorname{ad} P))^* \cong \Omega^2(\Sigma, \operatorname{ad} P)$ given by $\mu(A) = F_A \in \Omega^2(\Sigma, \operatorname{ad} P)$. The identification is given by $\eta(\xi) = \langle \eta, \xi \rangle = -\int_\Sigma \operatorname{Tr}(\xi \wedge \eta)$ for $\xi \in \Omega^0(\Sigma, \operatorname{ad} P), \eta \in \Omega^2(\Sigma, \operatorname{ad} P)$.*
3. *If $0 \in \Omega^2(\Sigma, \operatorname{ad} P)$ is a regular value of the moment map, then symplectic reduction $\mu^{-1}(0)/\,\mathcal{G}(P)$ is symplectic at smooth points(irreducible connections).*

Proof. 1. We define

$$\omega_{\mathrm{AB}}(\xi_1, \xi_2) = -\int_\Sigma \operatorname{Tr}(\xi_1 \wedge \xi_2),$$

where the trace operator should be recognized as the Killing form if not on a matrix Lie group, and are going to show that it is a 2-form compatible with J_A and then a symplectic form. Locally we have that $\xi_i = \xi_i^x \mathrm{d}x + \xi_i^y \mathrm{d}y, i = 1, 2$ where $\xi_i^x, \xi_i^y : U \to \mathfrak{su}(2) \cong \mathbb{R}^3$. Therefore $\omega_{\mathrm{AB}}(\xi_1, \xi_2) = -\int_\Sigma \mathrm{Tr}(\xi_1^x \xi_2^y - \xi_1^y \xi_2^x) \mathrm{d}x \wedge \mathrm{d}y = -\omega_{\mathrm{AB}}(\xi_1, \xi_2)$, deriving that ω_{AB} is a 2-form.

Recall that by compatible we mean that (1) $\omega_{\mathrm{AB}}(J_{\mathcal{A}}-, J_{\mathcal{A}}-) = \omega_{\mathrm{AB}}(-,-)$; and that (2) $\omega_{\mathrm{AB}}(-, J_{\mathcal{A}}-)$ is positively definitive. The former statement is obvious. The latter is none other than checking that $\omega_{\mathrm{AB}}(\xi_1, J_{\mathcal{A}}\xi_2) = -\int_\Sigma \mathrm{Tr}(\xi_1 \wedge *\xi_2) = \langle \xi_1, \xi_2 \rangle$ is exactly the L^2-inner product on $T_A \mathcal{A} = \Omega^1_{L^2_1}(\Sigma, \mathrm{ad}\, P)$.

The last thing is to show that ω is closed, i.e. $\mathrm{d}\omega(-,-,-) = 0$.

$\mathcal{G}(P)$ acts on $\mathcal{A}$ preserving ω_{AB}. Let $\mathcal{G}(P) \ni \phi : T_A \mathcal{A} \to T_A \mathcal{A} = \Omega^1(\Sigma, \mathrm{ad}\, P)$: $\xi[\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}(A_t)] \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}(\phi A_t \phi - \mathrm{d}\phi \cdot \phi^{-1}) = \mathrm{Ad}_\phi(\xi)$, we derive $\omega_{\mathrm{AB}}(\phi \cdot \xi_1, \phi \cdot \xi_2) = \omega_{\mathrm{AB}}(\xi_1, \xi_2)$.

2. Hamilton action is amount to say that each $\xi \in \mathrm{Lie}\, \mathcal{G}(P) = \Omega^0(\Sigma, \mathrm{ad}\, P)$ generates a vector field on $\mathcal{A}$, i.e. $\tilde{\xi}_A = \mathrm{d}_A \xi$ is a Hamiltonion vector field. That is, we need to check $D|_A(-\int_\Sigma \mathrm{Tr}(F_\bullet \wedge \xi))(\eta) = \omega_{\mathrm{AB}}(-\mathrm{d}_\bullet \xi, \eta)$ for $\xi \in T_A \mathcal{A} = \Omega^1(\Sigma, \mathrm{ad}\, P)$ as follows:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \left(\int_\Sigma - \mathrm{Tr}(F_{A+t\eta} \wedge \xi) \right) \\
=&- \int_\Sigma \mathrm{Tr}(\mathrm{d}_A \eta \wedge \xi) \\
=&- \int_\Sigma \mathrm{Tr}(\eta \wedge \mathrm{d}_A \xi) \\
=&- \int_\Sigma \mathrm{Tr}((-\mathrm{d}_A \xi) \wedge \eta) \\
=&\omega_{\mathrm{AB}}(\tilde{\xi}_A, \eta).
\end{aligned}$$

□

We denote as $\mathfrak{M}_\Sigma^{flat}(P)$ the moduli space of flat connections on a bundle P over a Riemann surface Σ. For a principal $\mathrm{SU}(2)$-bundle, we have $\dim \mathfrak{M}_\Sigma^{flat}(P) = 6g - 6 = (2g-2) \cdot 3$, where $2g - 2$ is the negetive of Euler's characteristic of Σ and 3 is the dimension of $\mathrm{SU}(2)$. Other than via A-S index theorem, the calculation can be done directly. Note that $\mathfrak{M}_\Sigma^{flat}(\Sigma \times G) = \dfrac{\hom(\pi_1(\Sigma_g), G)}{\sim}$ where $\sim$ denotes the G-conjugate action. Since $\pi_1(\Sigma_g) = \{a_1, \cdots, a_g, b_1, \cdots, b_g : \prod_{i=1}^d [a_i, b_i] = 1\}$, each generator can represent an element on the 3-dim group. The conjugate action is also of 3 dim. Therefore the dimension of moduli space is given by $3(2g-1) - 3 = 6g - 6$.

Suppose A is flat. We have an exact sequence

$$S : 0 \longrightarrow \Omega^0(\Sigma, \mathrm{ad}\, P) \xrightarrow{\mathrm{d}_A} \Omega^1(\Sigma, \mathrm{ad}\, P) \xrightarrow{\mathrm{d}_A} \Omega^2(\Sigma, \mathrm{ad}\, P) \longrightarrow 0.$$

Since $0 = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(F_{A_t}) = \mathrm{d}_A\eta$ for $A_t = A + t\eta$ a family of flat connections, we have at a smooth point $[A]$,

$$T_{[A]}\mathfrak{M}^{flat} = \{\eta|\mathrm{d}_A^*\eta = 0, \mathrm{d}_A\eta = 0\} = H^1(S),$$

which is a 1st order nonlinear elliptic PDEs. If the connection is irreducible, then H^0 vanishes. If further $\Omega^1 \to \Omega^2$ is surjective, then H^2 vanishes as well. We can determined the dimension of moduli space by applying Atiyah-Singer or Riemann-Roch to S.

Cohomological field theory.

Symplectic vortex equation. Suppose we have a principal G-bundle $P \longrightarrow \Sigma$ with connection A as well as a symplectic G-manifold (X, ω, J_X). We can define the associated bundle $P \times_G X \longrightarrow \Sigma$ with a section s on it. The symplectic vortex equation is given by

$$\begin{cases} \bar{\partial}_{A,J}s = 0(J\text{-holomorphic section}) \\ *_\Sigma F_A = \mu(s) \end{cases},$$

leading to the Hamiltonion Gromov-Witten invariant.

11 4D Gauge Theory (ASD Yang-Mills Theory)

Let $P \longrightarrow X^4$ be a principal SU(2)-bundle over a closed smooth Riemannian manifold X. Note that $c_1(P) = [\frac{\mathrm{i}}{2\pi}\operatorname{Tr} F_\nabla] = 0$ since locally we have $F^\alpha \in \Omega(U_\alpha, \mathfrak{su}(2))$ and $\mathfrak{su}(2) = \{\xi : \bar{\xi}^T + \xi = 0, \operatorname{Tr}\xi = 0\}$. We denote the **instanton number** by $k = \langle c_2(P), [X]\rangle = \frac{1}{8\pi^2}\int_X \operatorname{Tr}(F_A \wedge F_A)$ where $[X]$ is the fundamental class of X.

Notice that $(*|_{\Omega^2})^2 = \mathrm{Id}$, we have the eigenspace decomposition

$$\Omega^2(X, \operatorname{ad} P) = \Omega^2_+(X, \operatorname{ad} P) \oplus \Omega^2_-(X, \operatorname{ad} P)$$

corresponding to eigenvalue $1, -1$, respectively. There we have a self-dual and anti-self-dual decomposition

$$F_A = F_A^+ + F_A^-$$

where

$$F_A^\pm = \frac{F_A \pm *F_A}{2}.$$

Hence,

$$\mathrm{YM}(A) = -\int_X \operatorname{Tr}((F_A^+ + F_A^-) \wedge (F_A^+ - F_A^-)) = -\int_X \operatorname{Tr}(F_A^+ \wedge F_A^+) + \int_X \operatorname{Tr}(F_A^- \wedge F_A^-),$$

$$k = \frac{1}{8\pi^2}\int_X (\operatorname{Tr}(F_A^+ \wedge F_A^+) + \operatorname{Tr}(F_A^- \wedge F_A^-)).$$

Case 1. If $F_A^+ = 0$, i.e. F_A is anti-self-dual, there holds $\mathrm{YM}(A) = 8\pi^2 k > 0$.

Case 2. If $F_A^- = 0$ and $k < 0$, there holds $\mathrm{YM}(A) = -8\pi^2 k$.

Definition 11.1. *A critical point of* YM *on* $\mathcal{A}(P)$ *is called a* ***Yang-Mills connection***.

Proposition 11.2. *Yang-Mills connections contain those* A *satisfied* $\nabla_A^* F_A = -*\nabla_A * F_A = 0$.

Proof. For all $\xi \in T_A\mathcal{A} = \Omega^1(X, \operatorname{ad} P)$, we have $F_{A+t\xi} = F_A + t\nabla_A\xi + t^2\xi\wedge\xi$. Therefore,

$$\begin{aligned}
0 = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(\mathrm{YM}(A+t\xi)) &= -\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(\int_X \operatorname{Tr}(F_{A+t\xi} \wedge *F_{A+t\xi})\right) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(\langle F_{A+t\xi}, F_{A+t\xi}\rangle_{L^2}\right) = 2\langle \nabla_A\xi, F_A\rangle = 2\langle \xi, \nabla_A^* F_A\rangle,
\end{aligned}$$

leading to $\nabla_A^* F_A = 0$. □

Remark 11.3. $\nabla_A^* F_A = 0$ *is a 2nd order nonlinear PDE for* A.

Since it holds the bianchi identity that $\nabla_A F_A = 0$, connections with (anti-)self-dual curvature are automatically Yang-Mills connections.

Assume the instanton number $k > 0$ and note that

$$\mathrm{YM}(A) = \int_X (|F_A^+|^2 + |F_A^-|^2)\,\mathrm{dVol}_X,$$

$$8\pi^2 k = \int_X (-|F_A^+|^2 + |F_A^-|^2)\,\mathrm{dVol}_X\,.$$

It is easy to find out that $\mathrm{YM}(A) \geq 8\pi^2 k$, the equality holds iff $F_A^+ = 0$. We conclude that $\mathrm{YM}(-)$ achieves the absolute minimum at anti-self-dual Yang-Mills connections.

Remark 11.4. *$F_A^{\pm} = 0$ is 1st order PDE for A.*

We then consider the moduli space of anti-self-dual Yang-Mills connections with instanton number k, namely

$$\mathfrak{M}_k = \bigsqcup_{P : k = \langle c_2(P), [X] \rangle} \{A \in \mathcal{A}(P) \mid F_A^+ = 0\} / \mathcal{G}(P).$$

Firstly we introduce the Deformation complex, a.k.a. Atiyah-Hitchin-Singer complex, at an anti-self-dual connection A:

$$0 \longrightarrow \Omega^0(X, \mathrm{ad}\,P) \xrightarrow{\nabla_A} \Omega^1(X, \mathrm{ad}\,P) \xrightarrow{P_+ \circ \nabla_A} \Omega^2(X, \mathrm{ad}\,P) \longrightarrow 0.$$

Since $F_A^+ = P_+ \circ \nabla_A \circ \nabla_A = 0$, the above mappings indeed give rise to a complex.

Assume that $\mathfrak{M}_k$ is smooth at $[A]$, which requires A to be irreducible(i.e. $\mathcal{G}_A = \mathbb{Z}_2 \iff H_A^0 = \mathrm{Ker}(\mathrm{d}_A : \Omega^0(X, \mathrm{ad}\,P) \to \Omega^1(X, \mathrm{ad}\,P)) = 0$ since $\mathrm{Lie}\,\mathcal{G}(P) = \Omega^0(X, \mathrm{ad}\,P)$).

Theorem 11.5. *For a generic metric g on X (a Baire set in the space of all metrics), $H_A^2 = 0 \iff P_+ \circ \nabla_A : \Omega^1(X, \mathrm{ad}\,P) \to \Omega^2_+(X, \mathrm{ad}\,P)$ is surjective. Therefore, $\mathfrak{M}(P)$ is smooth at any irreducible anti-self-dual Yang-Mills connection.*

Essentially this is a ∞-dimensional implicit function theorem.

Theorem 11.6. *1. Let $\mathcal{C}$ be (the conformal class of) all Riemannian metric on X.*

$$\mathcal{E} = \bigsqcup_{(A,g)} \Omega^2_{+,g}(X, \mathrm{ad}\,P) \longrightarrow \mathcal{A}(P) \times \mathcal{C}$$

$$F^+ : \mathcal{A}(P) \times \mathcal{C} \to \mathcal{E} : (A, g) \mapsto F_g^+.$$

Then, $DF^+|_{(A,g)} : T_{(A,g)}(\mathcal{A} \times \mathcal{C}) \to \Omega^2_{+,g}(X, \mathrm{ad}\,P)$ is surjective. $(F^+)^{-1}(0)$ is a smooth manifold.

2. *For a generic metric g, $\mathfrak{M}_k^*$ is smooth and of dimension given by*

$$\begin{aligned}\dim H_A^1 =& \dim \operatorname{Ker}(P_+ \circ \nabla_A) \\ =& - \\ =& 8\left\langle c_2(P), [X]\right\rangle - 3(1 - b_1(X) + b_2^+(X)),\end{aligned}$$

where $b_1(X) = \dim H^1(X;\mathbb{R}), b_2^+(X) = \dim H_+^2(X;\mathbb{R})$.

$b_+^2(X) = 0$ iff the self intersection form $Q_X : H_2(X;\mathbb{Z}) \times H_2(X;\mathbb{Z}) \to \mathbb{Z}$ is negetively definite.

Theorem 11.7 (Donaldson)**.** *Let X be simply-connected with $b_2^+ = 0$. Let $k = 1$, for a generic metric, $\mathfrak{M}_1$ is 5-dim smooth manifold away from finitely many points $p_1, \cdots, p_{b_2}$, where $b_2 = b_2(X) = b_2^-(X)$. Moreover,*

1. *The neighbourhood of p_i is a cone over $\mathbb{C}P^2$.*

2. *$\mathfrak{M}_1^*$ is orientable.*

3. *(Taubes) There exists a diffeomorphism $(0, \varepsilon] \times X \hookrightarrow \mathfrak{M}_1^*$.*

4. *$\bar{\mathfrak{M}}_1 = \mathfrak{M}_1 \sqcup \{0\} \times X$, leading to that X is cobordant to $\#_{b_2}\mathbb{C}P^2$, and the self intersection form $Q_X \sim \begin{bmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{bmatrix}_{b_2 \times b_2}$.*

Corollary 11.8. *If Q_X is negatively/positively definite, then Q_X is diagnalizable. Therefore $\pm 2E_8$ has no smooth manifold realization. $Q_{K_3} = -2E_8 \oplus 3 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ leads to that $\mathbb{R}^4$ has an exotic C^∞-structure.*

Fact 11.9. *$\mathfrak{M}_1^*$ is non-empty.*

To show this fact, we are to write down an ASD connection on $\mathbb{R}^4$ or $\mathbb{S}^4$. First we identify $\mathbb{S}^4$ with $\mathbb{H}P^1 = \frac{\mathbb{H}^2 \setminus \{0\}}{\mathbb{H} \setminus \{0\}} = \frac{\mathbb{S}^7}{\mathrm{SU}(2)}$ since $\mathrm{SU}(2) = \{q \in \mathbb{H} : \|q\|^2 = 1\}$. Therefore we have a non-abelian Hopf fibration $\mathrm{SU}(2) \circlearrowright \mathbb{S}^7 \longrightarrow \mathbb{S}^4$. Note that $\mathbb{S}^4$ is a one-point compactification of $\mathbb{R}^4$. Since a point's difference does not affect the connection, we only need to define a connection on $\mathbb{R}^4$.

We define

$$A = \frac{\operatorname{Im}(x\mathrm{d}\bar{x})}{1 + |x|^2} \in \Omega^1(\mathbb{R}^4, \mathfrak{su}(2))$$

where $\mathfrak{su}(2) = \mathrm{Im}(\mathbb{H})$. By calculation

$$F_A = \mathrm{d}A + A \wedge A = \frac{\mathrm{d}x \wedge \mathrm{d}\bar{x}}{(1+|x|^2)^2}$$

is anti-self-dual and

$$\frac{1}{8\pi^2}\int_{\mathbb{R}^4} \|F_A\|^2 = 1$$

where we use the fact that $(\mathrm{d}x \wedge \mathrm{d}\bar{x}) \wedge *(\mathrm{d}x \wedge \mathrm{d}\bar{x}) = 48\,\mathrm{dVol}_{\mathbb{R}^4}$.

We then move the connection to a manifold. Note that when $x \to \infty$, $A \to 0$. Note also that every 4-dim manifold can be write as $X^4 = X^4 \# \mathbb{S}^4$. Therefore we can define a connection by first choosing a fibration on $\mathbb{S}^4$ with the connection defined above, and then composing it with trivial bundle with trivial connection over X.

12 Recent Works(WIP)

The moduli space $\mathfrak{M}_k$ has several compactifications. One of them given by Taubes and Bohui Chen is called *Bubble Tree Compactification*, which made $\bar{\mathfrak{M}}_k^B$ a smooth orbitfold. The orbitfold Euler characteristic class is an invariant, and one conjecture stated that it is equal to the Vafa-Witten invariant.

Donaldson's invariant. We have a map

$$\mu : H_2(X;\mathbb{Z}) \to H^2(\mathcal{B}^*),$$

$$\int_{\bar{\mathcal{B}}_b^k}^{Vir} \bigwedge_{i=1}^{d} \mu(\Sigma_i) = \text{Donaldson's invariant}.$$

Dimension reduction of ASD connections. Let $X = Y^3 \times \mathbb{R}$ with a trivial bundle $P = X \times \mathrm{SU}(2)$ over it. Since $\mathcal{A}(P) = \theta + \Omega^1(X, \mathfrak{su}(2))$, letting $\mathbb{A} \in \Omega^1(X, \mathfrak{su}(2))$, we have $\mathbb{A} = A(t) + A_0(t)\mathrm{d}t$, where $A(t) \in \Omega^1(X, \mathfrak{su}(2)), A_0(t) \in \Omega^0(X, \mathfrak{su}(2))$. Choose a gauge transform $g : Y \times \mathbb{R} \to \mathrm{SU}(2)$ such that

$$\frac{\partial g(t)}{\partial t} \cdot g^{-1}(t) = A_0(t)$$

to eliminate $A_0(t)\mathrm{d}t$ in $\mathbb{A}$, which is called in temporal gauge.

$$\begin{aligned} F_{\mathbb{A}} =& \mathrm{d}_{Y\times\mathbb{R}} A(t) + A(t) \wedge A(t) \\ =& \mathrm{d}_Y A(t) + A(t) \wedge A(t) + \mathrm{d}t \wedge \frac{\partial A(t)}{\partial t} \\ =& F_{A(t)} + \mathrm{d}t \wedge \frac{\partial A(t)}{\partial t} \end{aligned}$$

where $F_{A(t)}$ is the curvature of $A(t)$ as a connection on $Y \times \mathrm{SU}(2)$.

$$\begin{aligned} *_4 F_{\mathbb{A}} =& (*_3 F_{A(t)}) \wedge \mathrm{d}t - *_3 \frac{\partial A(t)}{\partial t} \\ =& - F_{\mathbb{A}} = -F_{A(t)} + \frac{\partial A(t)}{\partial t} \wedge \mathrm{d}t, \end{aligned}$$

deriving that

$$*_3 F_{A(t)} = \frac{\partial A(t)}{\partial t}.$$

Recall that

$$\nabla \, \mathrm{CS}(A) = - *_3 F_A,$$

we can summary that the ASD Yang-Mills equation on $P \longrightarrow Y \times \mathbb{R}$ in temporal gauge is the downward gradient flow equation of the Chern-Simons equation on $\mathcal{A}(P_Y \longrightarrow Y)$. The term *dimension reduction* means that if $A(t)$ is constant in t, we have that $F_A \equiv 0$ is a flat connection, reducing to 3D Chern-Simons theory.

Floer applied the idea of Witten's interpretation of Morse theory to develope the instanton homology theory for any integer homology 3-sphere(where all flat connection is either irreducible or trivial) which is $\mathbb{Z}_8$-graded and denoted by $HF_*^{\mathrm{YM}}(Y)$.

For more genera case, Fukaya for all admissible $\mathrm{SO}(3)-$bundle P over any 3-manifold Y with Stiefel-Whitney class $w_2(P) \neq 0$.

Remark 12.1. *If $w_2(P) = 0$, then $P = P_{\mathrm{SU}(2)}/\mathbb{Z}_2$. We have a lifting to $\mathrm{SU}(2)$ $\tilde{g}_{\alpha\beta}$ of $g_{\alpha\beta}$ such that*

$$\begin{tikzcd} & \mathrm{SU}(2) \arrow[d, "\mathbb{Z}_2"] \\ U_\alpha \cap U_\beta \arrow[ur, "\tilde{g}_{\alpha\beta}"] \arrow[r, "g_{\alpha\beta}"'] & \mathrm{SO}(3) \end{tikzcd} .$$

However, $\tilde{g}_\bullet$ does not satisfy the cocycle condition. In fact we have

$$\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma} = \varepsilon_{\alpha\beta\gamma}\tilde{g}_{\alpha\gamma}$$

where $\varepsilon_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \to \mathbb{Z}_2$. We call $\varepsilon_\bullet$ the Čech representation of $w_2(P)$.

We illuminate the idea of Fukaya-Floer homology. We have a complex

$$\bigoplus_{\langle a\rangle \in \mathfrak{M}^*_{flat}(P_Y)} \mathbb{Z}\langle a\rangle$$

with a boundary mapping

$$\partial : \langle a\rangle \mapsto \sum_\beta n_{\alpha\beta}\langle\beta\rangle ,$$

where $n_{\alpha\beta} = \#(\mathfrak{M}(\alpha,\beta)/\mathbb{R})^{[0]}$, where the superscript means the 0-dim components.

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