A Summary on Morse Theory

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April 10, 2022

Contents

This summary is aimed to give an introduction to Morse theory as well as some application in topology and some later development. I assume that the reader has been familiar with some basic knowledge of Riemannian geometry.

1 Morse Function on a Manifold

1.1 Morse Function

For a n-manifold M together with a smooth map

$$f : M \to \mathbb{R},$$

we call f is critical at x if

$$\mathrm{d}f_x = 0,$$

and non-degenerate if $\operatorname{Hess} f_x$ is non-singular, i.e.

$$\det\left(\operatorname{Hess} f_x\right) \neq 0.$$

We call a function is of **Morse** if all of its critical points are non-degenerated. Of course we do not know *a priori* if there exists a Morse function for a given manifold. However, in next section we will see not only that there exists a Morse function for any manifold, but also that almost every function is of Morse. We define the **index** of a non-degenerated critical point x as the dimension of maximal negetively definite subspace of $\operatorname{Hess} f_x$. From elementary linear algebra we know that it is also equal to the number of negative eigenvalues with multiplicity. More precisely, for a Morse function $f : M \to \mathbb{R}$,

Theorem 1.1. *Near each critical point of f, there exists a coordination $(x^1, \cdots, x^n)$ such that there holds the expression*

$$f = const - (x^1)^2 - (x^p)^2 + (x^{p+1})^2 + \cdots + (x^n)^2,$$

where p is the index of the critical point.

Therefore,

Corollary 1.2. *All critical points of a Morse function are isolated.*

The idea of Morse Theory is that the critical points of a Morse function can tell us the homotopy type of the manifold. To illustrate the idea rigorously, we consider the following facts. First we denote $M^a = f^{-1}(-\infty, a]$.

Theorem 1.3. *If $f^{-1}[a,b]$ is compact and contains no critical points, then there is a diffeomorphism $M^b \approx M^a$. Furthermore, M^a is deformation retract of M^b.*

Since all critical points are isolated, we can consider the level set passing through a critical point, which give rise to attaching a new cell.

Theorem 1.4. *Let the non-degenerated critical point be p with index λ, and $f(p) = c$. If $f^{-1}[c-\varepsilon, c+\varepsilon]$ is compact and contains no critical points other than p for some $\varepsilon > 0$, then $M^{c+\varepsilon}$ has the homotopy type of $M^{c-\varepsilon}$ with a λ-cell attached.*

After some preparation we could sum up the above idea as

Theorem 1.5. *If M^a is compact for each a, then M has the homotopy type of a CW-complex, with one cell of dimension λ for each critical point of index λ.*

1.2 Existence of a Morse Function(WIP)

Theorem 1.6 (Sard)**.** *If $f : M_1 \longrightarrow M_2$ is of C^1, where M_1, M_2 are differentiable manifold of the same dimension, then $f(\mathrm{crit}(f))$ is zero-measured in M_2.*

1.3 Morse's Inequality

The topology of M gives some constraints on the critical points of a Morse function by some inequalities which we will study in the following. Suppose M is compact.

Let C_λ denote the number of critical points of index λ, $b_\lambda(X,Y)$ the λ-th Betti number of (X,Y), $b_\lambda(M) = b_\lambda(M, \varnothing)$.

Theorem 1.7 (Weak Morse Inequality)**.** *$b_\lambda(M) \le C_\lambda$, and $\sum(-1)^\lambda b_\lambda(M) = \sum(-1)^\lambda C_\lambda$.*

Theorem 1.8 (Strong Morse Inequality)**.**

$$b_\lambda(M) - b_{\lambda-1}(M) + \cdots \pm b_0(M) \le C_\lambda - C_{\lambda-1} + \cdots \pm C_0$$

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1.4 Applications

Theorem 1.9 (Reeb)**.** *A n-manifold with only 2 critical points is topologically a sphere $\mathbb{S}^n$.*

Theorem 1.10. *$\mathbb{C}P^n$ has the homotopy type of $e^0 \cup e^2 \cup \cdots \cup e^{2n}$.*

Therefore, from cellular homology we derive that $H_i(\mathbb{C}P^n;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 0, 2, \cdots, 2n \\ 0, & \text{otherwise} \end{cases}$.

2 Topology of Path Space

2.1 Path Space and Energy Functional

Suppose M is a complete Riemannian n-manifold with metric $g = \langle -, - \rangle$. Let $\Omega(M; a, b)$ denote all piecewise smooth path connecting from a to b in M. It is obvious that the space is infinitly dimensional. The tangent space $T_\omega \Omega(M; a, b)$ of $\omega(M; a, b)$ at ω consists of piecewise smooth vector fields such that $W(0) = W(1) = 0$.

We define the energy functional as

$$E : \Omega(M; a, b) \to \mathbb{R} : \gamma \mapsto \int_0^1 \|\dot{\gamma}(t)\|^2 \, \mathrm{d}t,$$

where $\gamma(0) = a, \gamma(1) = b$. We claim that E is a Morse function on the infinitly dimensional space $\Omega(M; a, b)$. To show this we need to study its critical points and Hessian.

Let $\bar{\alpha}(u) = \alpha(u; t)$ be 1-parameter variation of γ with variation vector field $W_t = \left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_{u=0} \bar{\alpha}_t$, then we think of the differential at γ to be

$$\mathrm{d}E_\gamma : T_\gamma \Omega(M; a, b) \to \mathbb{R} : W \mapsto \left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_{u=0} E(\bar{\alpha}).$$

Therefore, a critical path γ of E should satisfy $\mathrm{d}E_\gamma = 0$. We will not give the conditions that make the definition be well-defined, neither will show it. Instead, we define a path to be critical iff for all variation α there holds $\left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_{u=0} E(\bar{\alpha}) = 0$.

Similarly, let $\bar{\beta}(u_1, u_2) = \beta(u_1, u_2; t)$ be a 2-parameter variation with $\beta(0, 0; t) = \gamma(t)$, $\frac{\partial \beta}{\partial u_1}(0, 0; t) = W_1(t)$, $\frac{\partial \beta}{\partial u_2}(0, 0; t) = W_2(t)$. We claim that the Hessian of E at a critical path γ is given by

$$\operatorname{Hess} E_\gamma(W_1, W_2) = \left.\frac{\partial^2}{\partial u_1 \partial u_2}\right|_{(0,0)} E(\bar{\beta}).$$

We will show it is exactly a well-defined quadratic form. *A priori* we do not know the index of such a quadratic form is meaningful because it could be infinity. However, in next subsection we will see that the index of E at a critical path is surprisingly always finite.

So if $\Omega(M; a, b)$ is a manifold, then by the theory from the last section we can conclude that $\Omega(M; a, b)$ has the homotopy type of a finite CW complex, with one cell of dimension λ for each critical point of index λ. However, it is not true that $\Omega(M; a, b)$ is a manifold; however, the conclusion remains true. We can use some finite dimensional manifold to approximate $\Omega(M; a, b)$, which we will investigate in the following subsections.

In order to do calculation, we make some definition and evaluate two useful formula then. Let $V_t = \dot\omega$ be the velocity vector of ω, $A_t = \nabla_{\dot\omega}\dot\omega$ the acceleration vector, $\Delta_t V = V_{t+} - V_{t-}$ the discontinuity in the velocity vector at t. We have

Theorem 2.1 (First variation formula of E)**.**

$$\frac{1}{2}\left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_{u=0} E(\bar\alpha) = -\sum_t \langle W_t, \Delta_t V\rangle - \int_0^1 \langle W_t, A_t\rangle\,\mathrm{d}t.$$

From the formula we easily derive that

Corollary 2.2. *γ is a critical path iff it is a geodesic.*

So we only need to focus on geodesics.

Theorem 2.3 (Second variation formula of E)**.**

$$\frac{1}{2}\left.\frac{\partial^2}{\partial u_1 \partial u_2}\right|_{(0,0)} E(\bar\beta) = -\sum_t \langle W_2, \Delta_t \nabla_{\dot\gamma} W_1\rangle - \int_0^1 \langle W_2, \nabla_{\dot\omega}\nabla_{\dot\omega} W_1 + R(V, W_1)V\rangle\,\mathrm{d}t.$$

Immediately,

Corollary 2.4. $\operatorname{Hess} E$ *is a well-defined symmetric and bilinear functional of W_1 and W_2.*

Moreover,

Theorem 2.5. *W belongs to the null space of* $\operatorname{Hess} E$ *iff W is a Jacobi field. Hence E is degenerate iff a, b are conjugate points, with nullity equal to multiplicity of a, b as conjugate points.*

Non-conjugate points always exist. In fact we have

Lemma 2.6. $\exp_p v$ *is conjugate to p along a geodesic iff* $\exp_p$ *is critical at v.*

Therefore, together with Sard's theorem, we get that for a fixed p, almost all q is not conjugate to p along any geodesic.

2.2 Index Theorem

Theorem 2.7 (Morse)**.** *The index* $\operatorname{ind}\operatorname{Hess} E$ *is equal to number of points $\gamma(t)(0 < t < 1)$ such that $\gamma(t)$ is conjugate to $\gamma(1)$ with multiplicity. Moreover,* $\operatorname{ind}\operatorname{Hess} E < \infty$.

To prove it, we need the following definitions and results.

Lemma 2.8.

$$T_\gamma\Omega = T_\gamma\Omega(t_0, \cdots, t_k) \oplus T'.$$

Lemma 2.9. $\operatorname{ind}\operatorname{Hess} E = \operatorname{ind}\operatorname{Hess} E|_{T_\gamma\Omega(t_0,\cdots,t_n)}$.

2.3 Topology

First we let the distance function

$$d(\omega, \omega') = \max_{0\le t\le 1} \rho(\omega(t), \omega'(t)) + \left[\int_0^1 \left(\frac{\mathrm{d}s}{\mathrm{d}t} - \frac{\mathrm{d}s'}{\mathrm{d}t}\right)^2 \mathrm{d}t\right]^{1/2}$$

induces the topology on Ω. Then E is continuous. Therefore $\Omega^c := E^{-1}[0, c]$ is a closed subset. Ω^c has a finite dimensional approximation. Notice

Theorem 2.10. *Fix $c > 0$ such that $\Omega^c \neq \varnothing$. For all sufficiently fine subdivisions $(t_0, \cdots, t_k)$ of $[0, 1]$, $B := \operatorname{Int}\Omega(t_0, \cdots, t_k)^c$ can be given the structure of a smooth finite dimensional manifold in a natural way.*

Such B is what we need to study Ω^c.

Theorem 2.11. *B^a is compact and is a deformation retract of Ω^a. The critical points are the same. The index of the same critical point is also the same.*

To sum up,

Theorem 2.12. *If p, q are not conjugate along any geodesic of length $\le \sqrt{a}$, then Ω^a has the homotopy type of a finite CW-complex, with one cell of dimension λ for each geodesic in Ω^a at which* $\operatorname{ind}\operatorname{Hess} E = \lambda$.

To study Ω, we need definie Ω^* of all continuous path connecting from a to b with compact open topology induced by

$$d^*(\omega, \omega') = \max_t \rho(\omega(t), \omega'(t)).$$

We have that

Theorem 2.13. *$i : \Omega \to \Omega^*$ is a homotopy equivalence.*

Also notice that Ω^* has the homotopy type of a CW-complex, hence so does Ω. Moreover,

Theorem 2.14 (Fundamental theorem of Morse theory.)**.** *If a, b are not conjugate along any geodesic, then Ω has the homotopy type of a countable CW-complex which contains one cell of dimension λ for each geodesic from a to b of index λ.*

2.4 Loop Space and Freudenthal Suspension Theorem

By perturbation we have a homotopy equivalence $\Omega(M; p, q) \simeq \Omega(M; p, p)$ on a complete Riemannian manifold. We call the latter space the loop space of M and denote it by ΩM. In this subsection we mainly focus on the case of $M = \mathbb{S}^n$. By counting multiplicity of conjugate points on geodesics, we derive

Theorem 2.15. *$\Omega\mathbb{S}^n$ has the homotopy type of a CW-complex with one cell each in the dimensions $0, n-1, 2(n-1), \cdots$.*

Let $\Omega^{\pi^2} = \Omega^{\pi^2}(\mathbb{S}^{n+1}; p, -p)$ denote the space of minimal geodesics between two antipodal points. Since there is a 1-to-1 correspondence between minimal geodesics of $\mathbb{S}^{n+1}$ and the points in the equator $\mathbb{S}^n \subset \mathbb{S}^{n+1}$, we will see $\Omega^{\pi^2}\mathbb{S}^{n+1}$ is a good object to represent $\mathbb{S}^n$. Following such an idea, we consider $p, q \in M$ with $\rho(p, q) = \sqrt{d}$.

Theorem 2.16. *If Ω^d is a manifold and if every non-minimal geodesic from p to q has index $\geq \lambda_0$, then*

$$\pi_i(\Omega, \Omega^d) = 0$$

for $0 \leq i < \lambda_0$. Therefore

$$\pi_i\Omega^d \cong \pi_i\Omega \cong \pi_{i+1}M$$

for $0 \leq i \leq \lambda_0 - 2$.

Directly,

Corollary 2.17 (Freudenthal suspension theorem)**.**

$$\pi_i\mathbb{S}^n \cong \pi_{i+1}\mathbb{S}^{n+1}$$

for $i \leq 2n - 2$.

3 Bott's Periodicity Theorems

3.1 The Unitary Group

Let $\mathrm{SU}(2m)$ be a Riemannian manifold with a left and right invariant metric $\langle -, - \rangle$, namely

$$\langle A, B\rangle := \operatorname{Re}\operatorname{Tr}(AB^*) = \operatorname{Re}\sum A_{ij}\bar{B}_{ij}$$

for $A, B \in \mathfrak{su}(2m)$. We are going to consider the set of all geodesics in $\mathrm{SU}(2m)$ from I to $-I$, i.e. looking for $A \in \mathfrak{u}(2m)$ such that $\exp A = -I$. Let TAT^{-1} be in diagonal form. Since $\exp(TAT^{-1}) = T(\exp A)T^{-1} = -I$, we can assume A is always diagonal, i.e.

$$A = \begin{bmatrix} \mathrm{i}\, a_1 & & \\ & \ddots & \\ & & \mathrm{i}\, a_{2m} \end{bmatrix},$$

and

$$\exp A = \begin{bmatrix} \mathrm{e}^{\mathrm{i}\, a_1} & & \\ & \ddots & \\ & & \mathrm{e}^{\mathrm{i}\, a_{2m}} \end{bmatrix}.$$

So if $\exp A = -I$ then there holds $a_i = k_i\pi$ where k_i's are all odd. The length of the geodesic $t \mapsto \exp At, 0 \leq t \leq 1$ is given by $\|A\| = \sqrt{\operatorname{Tr} AA^*} = \pi\sqrt{k_1^2 + \cdots + k_{2m}^2}$. Hence the minimal geodesics make k_i be ± 1. Note that in $\mathfrak{su}(2m)$ there holds $\operatorname{Tr} A = 0$, we derive that the number of k_i's being 1 is equal to the number of being -1, which means the eigenspace $\mathrm{Eigen}(-\mathrm{i}\,\pi)$ of eigenvalue $-\mathrm{i}\,\pi$ and the eigenspace $\mathrm{Eigen}(\mathrm{i}\,\pi)$ of eigenvalue $\mathrm{i}\,\pi$ have the same dimension. Thus A is completely determined by $\mathrm{Eigen}(\mathrm{i}\,\pi)$, which is an arbitrary m-subspace of $\mathbb{C}^{2m}$. To sum up,

Theorem 3.1. *There holds a homeomorphism*

$$\Omega^{\pi\sqrt{2m}}\,\mathrm{SU}(2m) \cong \mathrm{Gr}_m(\mathbb{C}^{2m}).$$

We will prove that

Theorem 3.2. *Every non-minimal geodesic from I to $-I$ in $\mathrm{SU}(2m)$ has index $\geq 2m+2$.*

Therefore,

Corollary 3.3 (Bott)**.**

$$\pi_i \operatorname{Gr}_m(\mathbb{C}^{2m}) \cong \pi_{i+1} \operatorname{SU}(2m)$$

for $i \leq 2m$.

In order to see the relationship with unitary group, we need some preparation. From the fibration

$$\operatorname{U}(m) \longrightarrow \operatorname{U}(m+1) \longrightarrow \mathbb{S}^{2m+1}$$

, we have an exact sequence

$$\cdots \longrightarrow \pi_i \mathbb{S}^{2m+1} \longrightarrow \pi_{i-1} \operatorname{U}(m) \longrightarrow \pi_{i-1} \operatorname{U}(m+1) \longrightarrow \pi_{i-1} \mathbb{S}^{2m+1} \longrightarrow \cdots,$$

hence the inclusion map becomes an isomorphism

$$\pi_{i-1} \operatorname{U}(m) = \pi_{i-1} \operatorname{U}(m+1)$$

for $i \leq 2m$. We call it the $(i-1)$-st **stable homotopy group** of the unitary group, denoted by $\pi_{i-1} \operatorname{U}$. The exact sequence also shows that $\pi_{2m} \operatorname{U}(m) \to \pi_{2m} \operatorname{U}(m+1) \cong \pi_{2m} \operatorname{U}$ is onto.

Another fibration

$$\operatorname{U}(m) \longrightarrow \operatorname{U}(2m) \longrightarrow \operatorname{U}(2m)/\operatorname{U}(m)$$

gives rise to

$$\pi_i(\operatorname{U}(2m)/\operatorname{U}(m)) = 0$$

for $i \leq 2m$.

Since $\operatorname{Gr}_m(\mathbb{C}^{2m}) = \operatorname{U}(2m)/\operatorname{U}(m) \times \operatorname{U}(m)$, there holds a fibration

$$\operatorname{U}(m) \longrightarrow \operatorname{U}(2m)/\operatorname{U}(m) \longrightarrow \operatorname{Gr}_m(\mathbb{C}^{2m}),$$

which derives

$$\pi_i \operatorname{Gr}_m(\mathbb{C}^{2m}) \cong \pi_{i-1} \operatorname{U}(m)$$

for $i \leq 2m$.

Finally, from the fibration

$$\operatorname{SU}(m) \longrightarrow \operatorname{U}(m) \longrightarrow \mathbb{S}^1,$$

we have

$$\pi_i \operatorname{SU}(m) \cong \pi_i \operatorname{U}(m)$$

for $i \neq 1$.

Above all, we see that

$$\pi_{i-1}\,\mathrm{U} = \pi_{i-1}\,\mathrm{U}(m) \cong \pi_i\,\mathrm{Gr}_m(\mathbb{C}^{2m}) \cong \pi_{i+1}\,\mathrm{SU}(2m) \cong \pi_{i+1}\,\mathrm{U}(2m) = \pi_{i+1}\,\mathrm{U}$$

for $1 \leq i \leq 2m$. Therefore

$$\pi_{i-1}\,\mathrm{U} \cong \pi_{i+1}\,\mathrm{U}$$

for $i > 0$. Since $\pi_0\,\mathrm{U} = \pi_0\,\mathrm{U}(1) \cong 0, \pi_1\,\mathrm{U} = \pi_1\,\mathrm{U}(1) \cong \mathbb{Z}$, we can conclude that

Theorem 3.4 (Bott's periodicity theorem for unitary group)**.**

$$\pi_0\,\mathrm{U} \cong \pi_2\,\mathrm{U} \cong \pi_4\,\mathrm{U} \cong \cdots \cong 0,$$

$$\pi_1\,\mathrm{U} \cong \pi_3\,\mathrm{U} \cong \pi_5\,\mathrm{U} \cong \cdots \cong \mathbb{Z}.$$

3.2 The Orthogonal and Symplectic Group

Definition 3.5. *A* ***complex structure*** *J on $\mathbb{R}^n$ is a linear transformation $J : \mathbb{R}^n \to \mathbb{R}^n$, belonging to the orthogonal group, which satisfies the identity $J^2 = -I$. The space is denoted as $\Omega_1(n)$.*

Next, we study $\mathrm{O}(n)$. Let $n = 2m$.

Theorem 3.6.

$$\Omega^m(\mathrm{O}(2m); I, -I) \cong \Omega_1(2m).$$

Theorem 3.7. *All non-minimal geodesic from I to $-I$ in $\mathrm{O}(2m)$ has index $\geq 2m-2$.*

Theorem 3.8 (Bott)**.**

$$\pi_i\Omega_1(2m) \cong \pi_{i+1}\,\mathrm{O}(2m)$$

for $i \leq 2m-4$.

Then we will iterate the above procedure, studying the space of geodesics from J to $-J$ in $\Omega_1(n)$; and so on. Assume that n is divisible by some power of 2.

Let $J_1, \cdots, J_k$ be mutually anti-commute fixed complex structure on $\mathbb{R}^n$. Suppose there exists at least one other complex structure J which anti-commute with J_i's. Let $\Omega_k(n)$ denote the set of all complex structure which anti-commute with the fixed structure $J_1, \cdots, J_k$. Then we have

$$\Omega_k(n) \subset \Omega_{k-1}(n) \subset \cdots \subset \Omega_1(n) \subset \mathrm{O}(n) =: \Omega_0(n).$$

$\Omega_k(n)$ is compact.

Theorem 3.9. *$\Omega_k(n)$'s are all smooth, totally geodesic submanifolds of* $\mathrm{O}(n)$*.*

$$\Omega^m(\Omega_l(n); J_l, -J_l) \cong \Omega_{l+1}(n)$$

for $0 \leq l < k$*.*

Let $\Omega_k = \lim_{n\to\infty} \Omega_k(n)$ and $\mathrm{O} = \Omega_0$. We call the latter **infinite orthogonal group**. And the inclusions $\Omega_{k+1}(n) \to \Omega\Omega_k(n)$ give rise to a inclusion $\Omega_{k+1} \to \Omega\Omega_k$. After studying the non-minimal geodesic on $\Omega_k(n)$, we will see that the index is $\geq n/m_{k+1} - 1$. Hence,

Theorem 3.10. *$\Omega_{k+1} \to \Omega\Omega_k$ is a homotopy equivalence. Thus we have isomorphisms*

$$\pi_h \mathrm{O} \cong \pi_{h-1}\Omega_1 \cong \cdots \cong \pi_1\Omega_{h-1}.$$

We now give descriptions of the manifolds $\Omega_k(n)$. We let $n = 16r$.

- $\Omega_0(n)$ is the orthogonal group.
- $\Omega_1(n)$ is the set of all complex structures on $\mathbb{R}^n$.
- $\Omega_2(n) = \mathrm{U}(n/2)/\mathrm{Sp}(n/4)$ is the set of quaternion structures on $\mathbb{C}^{n/2}$.
- $\Omega_3(16r)$ is the quaternionic Grassmann manifold of $\mathbb{H}^{4r}$.
- $\Omega_4(16r) = \mathrm{Sp}(2r)$ is the set of all quaternionic isometries from V_1 to V_2.
- $\Omega_5(16r) = \mathrm{Sp}(2r)/\mathrm{U}(2r)$ is the set of subspaces $W \subset V_1$ such that W is closed under J_1 and V_1 splits as the orthogonal sum $W \oplus J_2W$.
- $\Omega_6(16r) = \mathrm{U}(2r)/\mathrm{O}(2r)$ is the set of all real subspaces $X \subset W$ such that W splits as the orthogonal sum $X \oplus J_1X$.
- $\Omega_7(16r)$ is the real Grassmann manifold consisting of all real subspaces of $X \cong \mathbb{R}^{2r}$.
- $\Omega_8(16r) = \mathrm{O}(r)$ is the set of all real isometries from X_1 to X_2.

By passing to the limit as $r \to \infty$ we get $\Omega_8 \cong \mathrm{O}$. Therefore,

Theorem 3.11 (Bott)**.**

$$\pi_i \mathrm{O} \cong \pi_{i+8} \mathrm{O}$$

for $i \geq 0$*.*

Since $\mathrm{Sp} = \Omega_4$, we have $\mathrm{O} \cong \Omega^4 \mathrm{Sp}$ as well as $\mathrm{Sp} \cong \Omega^4 \mathrm{O}$. To conclude,

$i \bmod 8$	$\pi\_i\,\mathrm{O}$	$\pi_i\,\mathrm{Sp}$
0	$\mathbb{Z}_2$	0
1	$\mathbb{Z}_2$	0
2	0	0
3	$\mathbb{Z}$	$\mathbb{Z}$
4	0	$\mathbb{Z}_2$
5	0	$\mathbb{Z}_2$
6	0	0
7	$\mathbb{Z}$	$\mathbb{Z}$

4 Morse Homology(WIP)

In a Riemannian manifold (M, g), we let $V = -\nabla f$. We define a one-parameter group of diffeomorphisms $\Psi_s : M \to M$ for $s \in \mathbb{R}$ and $\frac{\mathrm{d}\Psi}{\mathrm{d}t} = V$. If p is a critical point, We define the descending manifold (unstable manifold) and ascending manifold (stable manifold) by

$$D(p) = \{x \in M : \lim_{s \to -\infty} \Psi_s(x) = p\}$$

and

$$A(p) = \{x \in M : \lim_{s \to \infty} \Psi_s(x) = p\},$$

respectively. If p is a non-degenerate critical point, then $D(p)$ is an embedded open disk in M with dimension $\dim D(p) = \operatorname{ind} p$, since $T_pD(p)$ is the negetive eigenspace of $\operatorname{Hess} f(p)$. Similarly, $A(p)$ is an embedded open disk with dimension $\dim A(p) = \dim M - \operatorname{ind} p$. We call a pair (f, g) is Morse-Smale if $D(p)$ is transverse to $A(q)$ for each pair of critical points p, q. Like Morse function, we will such condition holds generically.

We define Morse complex as follows. Let

$$C_i^M(f, g) := \mathbb{Z}[\mathrm{crit}_i(f)].$$

The differnetial counts the gradient flow lines, namely,

$$\partial^M(p) := \sum_{q \in \mathrm{crit}_{i-1}(f)} \#M(p, q) \cdot q.$$

The homology of Morse complex is isomorphic to singular homology of M:

$$H_*^M(f, g) \cong H_*(M; \mathbb{Z}).$$

5 h-Cobordism(WIP)

5.1 Smale Cancellation

References

[M1] J. Milnor: Morse Theory.

[M2] J. Milnor: Lectures On The h-Cobordism Theorem.